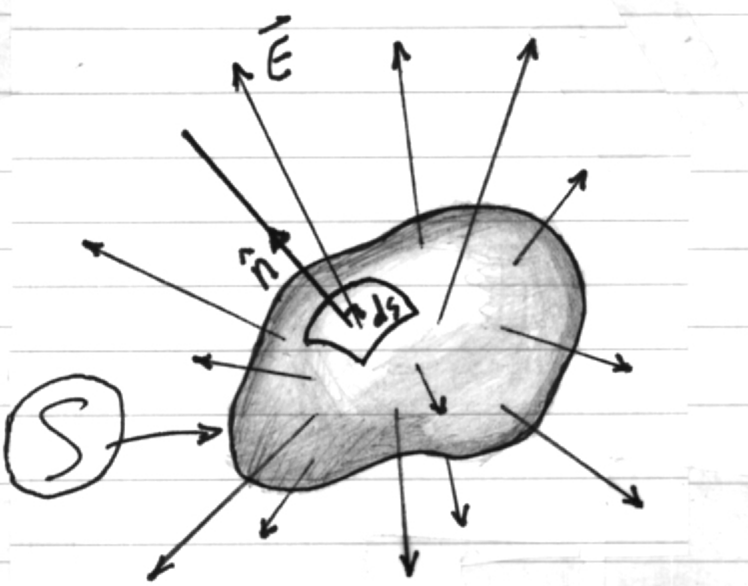
هندسة اتصالات الحاسبات
المرحلة الثالثة
نظرية المجالات

EM. 4

Gauss' Law

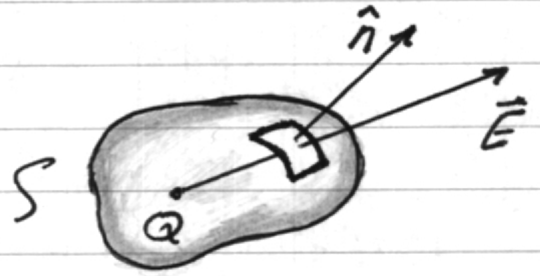
*- The integral of the electric field intensity ($\vec{E}$) over a closed surface is equal to the enclosed electric charge devided by the permittivity of the medium.



$$\boxed{\oint_s \vec{E}\cdot\vec{ds} = Q / \epsilon}$$

Integral form of Gauss' Law.

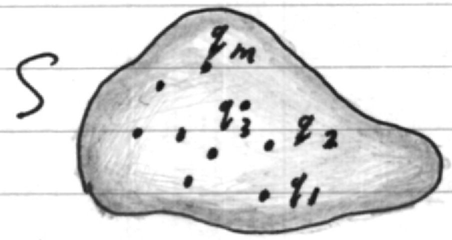
*- The closed surface over which the integration is performed is known as "The Gauss Surface".



A Point charge

$$\boxed{\oint_S \vec{E} \cdot \vec{ds} = Q / \epsilon}$$

System of point charges
or, Collection of point charges

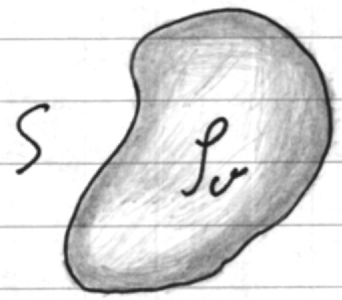


$$Q = \sum_{m=1}^{n} q_m$$

*- We have n - point charges enclosed by surface S.

$$\boxed{\oint_S \vec{E} \cdot \vec{ds} = \frac{1}{\epsilon} \sum_{m=1}^{n} q_m}$$

Volume charge distribution



*- We have a total charge Q, distributed throughout volume V surrounded by surface S.

$$Q = \int_V \rho_v \, dv$$

$$\therefore \boxed{\oint_S \vec{E} \cdot \vec{ds} = \frac{1}{\epsilon} \int_V \rho_v \, dv}$$

Differential Form of Gauss' Law

*- We have : $\oint_s \vec{E} \cdot \vec{ds} = Q / \epsilon$

*- Using the divergence theorem :

$$\oint_s \vec{E} \cdot \vec{ds} = \int_v \vec{\nabla} \cdot \vec{E} \, dv$$

*- Gauss' law becomes :

$$\int_v \vec{\nabla} \cdot \vec{E} \, dv = Q / \epsilon$$

*- If the charge (Q) in volume (V) has a volume distribution (ρ_v) $\{Q = \rho_v V\}$, then:

$$\int_v \vec{\nabla} \cdot \vec{E} \, dv = \int_v \frac{\rho_v}{\epsilon} \, dv$$

*- Comparing both sides, we get :

$$\boxed{\vec{\nabla} \cdot \vec{E} = \rho_v / \epsilon}$$ ← Differential form Gauss' Law

Electric Flux and Flux Density

Electric Flux

*- Electric charges may be viewed as if they emit an electric flux which is responsible for the effects of these charges at remote distances.

*- Electric flux (Ψ) is represented by the stream lines.

*- Other names for the electric flux : Displacement flux , Flux displacement.

*- The total flux (Ψ_T) for a given charge (Q) is equal to that charge ($\Psi_T = Q$).

*- The unit of the electric flux is "Coulomb" (C).

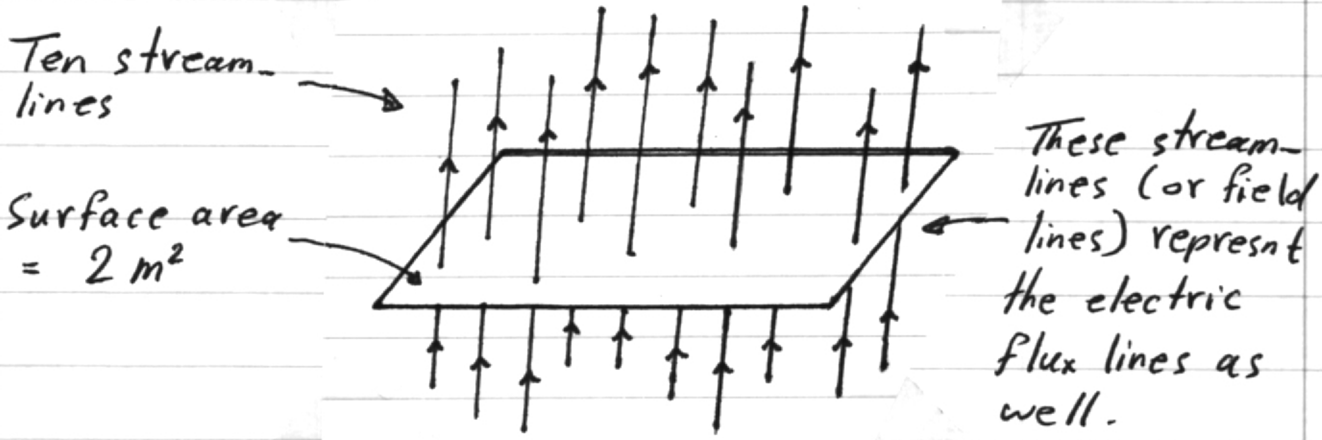
Electric Flux Density

*- A simple definition : Flux density is the flux per unit area ; Units : C/m^2.

*- Symbol : D , $D = \frac{\Psi}{S}$

*- Flux density is the amount of stream lines (or flux lines) crossing a given area perpendicularly devided by that area.

Illustration:

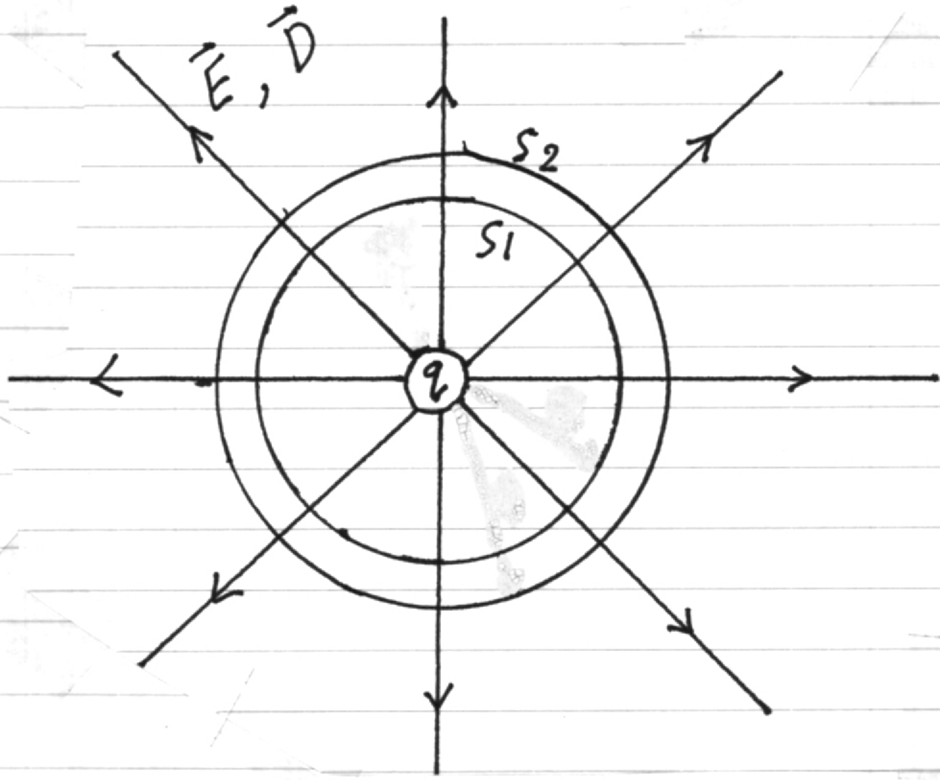


*- $\Psi = 10\ C$, $D = \frac{10}{2} = 5\ C/m^2$

*- The flux lines in the above figure cross the surface perpendicularly.

*- The flux density is a vector quantity ($\vec{D}$), and its direction is the same as that of the electric field intensity ($\vec{E}$).

Illustration :



*- $q = 8\ C$

$S1 = 2\ m^2$, $S2 = 4\ m^2$

*- Flux density on S1 : $\vec{D}_1 = \frac{8}{2}\hat{a}_r = 4\hat{a}_r\ C/m^2$.

*- Flux density on S2 : $\vec{D}_2 = \frac{8}{4}\hat{a}_r = 2\hat{a}_r\ C/m^2$

*- Flux on S1 : $\Psi_1 =$

Flux on S2 : $\Psi_2 =$

Total flux : $\Psi_T =$

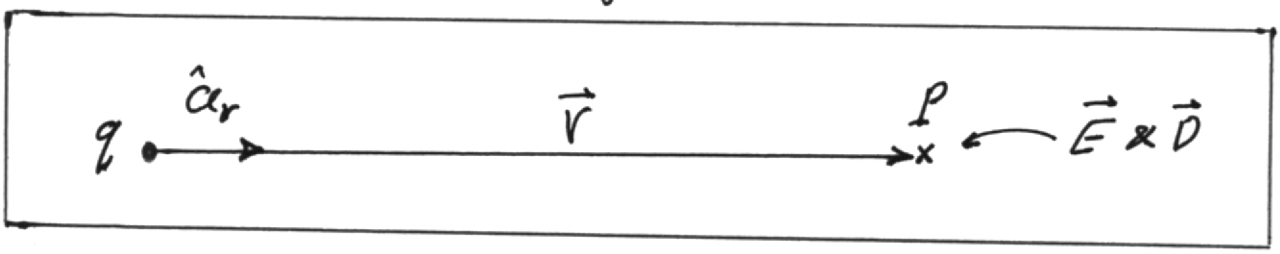
*- Other names for $\vec{D}$: *- Displacement Flux Density.

*- Displacement Flux.

Relation Between $\vec{D}$ and $\vec{E}$

$$\vec{D} = \epsilon \vec{E} \quad \text{—— (a)}$$

*- For a point charge (q), we may determine $\vec{E}$ and $\vec{D}$ at any point (P) at a distance (r) from that charge :



*- Magnitudes of $\vec{E}$ and $\vec{D}$

$$E = k \frac{q}{r^2}$$

or, $$E = \frac{q}{4\pi \epsilon r^2}$$

$$\boxed{D = \frac{q}{4\pi r^2}} \quad \text{—— (a')}$$

← For a point charge.

*- Directional $\vec{E}$ and $\vec{D}$

$$\vec{E} = E\,\hat{a}_r$$

$$\vec{E} = k\frac{q}{r^2}\,\hat{a}_r$$

$$\vec{E} = k\frac{q}{r^3}\,\vec{r}$$

$$\vec{D} = D\,\hat{a}_r$$

$$\vec{D} = \frac{q}{4\pi r^2}\,\hat{a}_r \quad \text{—— (b)}$$

$$\vec{D} = \frac{q}{4\pi r^3}\,\vec{r} \quad \text{—— (c)}$$

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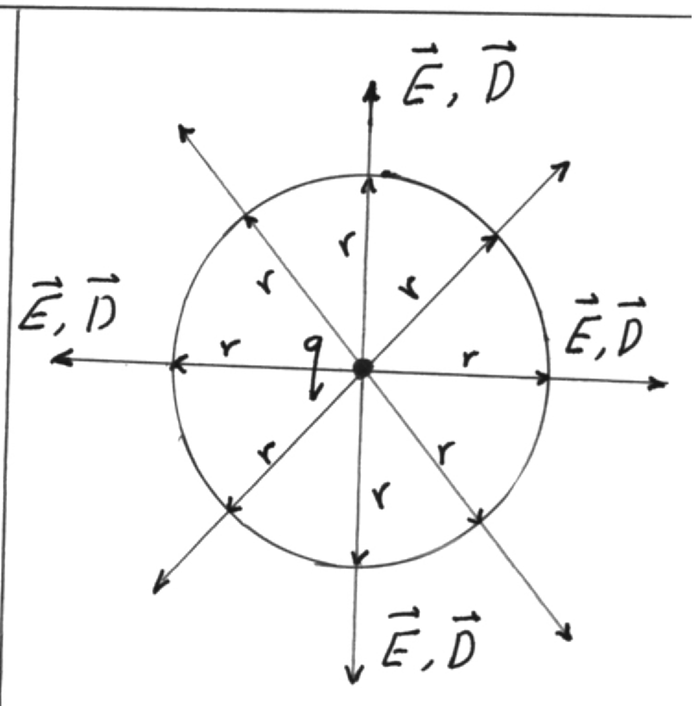
*- Eq. (a) : $D = \frac{q}{4\pi r^2}$ gives the amount of the flux density (D) for any point at a distance (r) from a point charge (q).

*- To include direction, a suitable unit vector is chosen { eq. (b) }.



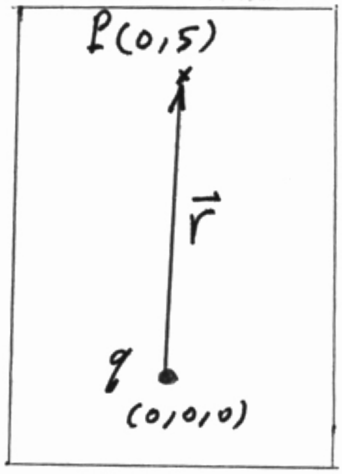
Ex.: A point charge $q = 400\ \mu C$ placed at the origin of an x-y coordinates. Determine:

1- The electric flux density $(\vec{D})$ at point $P(0,5)m$.

2- The electric field intensity $(\vec{E})$ at (P).

3- Use the electric field intensity to determine the force exerted on a point charge $Q = -18\ \mu C$ when placed at point (P).

{The Medium is Free Space: $\epsilon_o = 8.854 \times 10^{-12}\ F/m$}



Sol.:

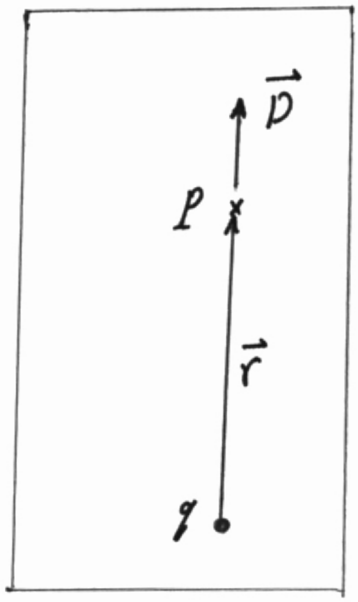
1- $$\vec{D} = \frac{q}{4\pi r^3}\ \vec{r}$$

$$\vec{r} = 5\hat{j}\ m \quad , \quad r = 5\ m$$

$$\vec{D} = \frac{400 \times 10^{-6}}{4 \times 3.14 \times 5^3}\ (5\hat{j})$$

$$\vec{D} = 1.2738 \times 10^{-6}\ \hat{j}\ C/m^2$$

$$\boxed{\text{or, } \vec{D} = 1.2738\ \hat{j}\ \mu C/m^2}$$



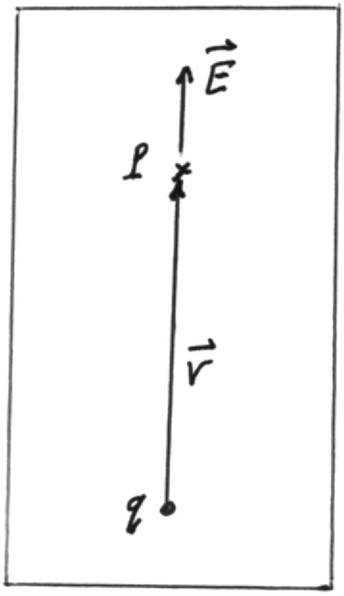
2- In free space : $\vec{D} = \epsilon_0 \vec{E}$

$$\therefore \vec{E} = \frac{\vec{D}}{\epsilon_0} = \frac{1.2738 \times 10^{-6}\ \hat{j}}{8.854 \times 10^{-12}}$$

$$\vec{E} = 0.1438 \times 10^{6}\ \hat{j}\ N/C$$

or, $\vec{E} = 0.1438\ \hat{j}\ M\,N/C$

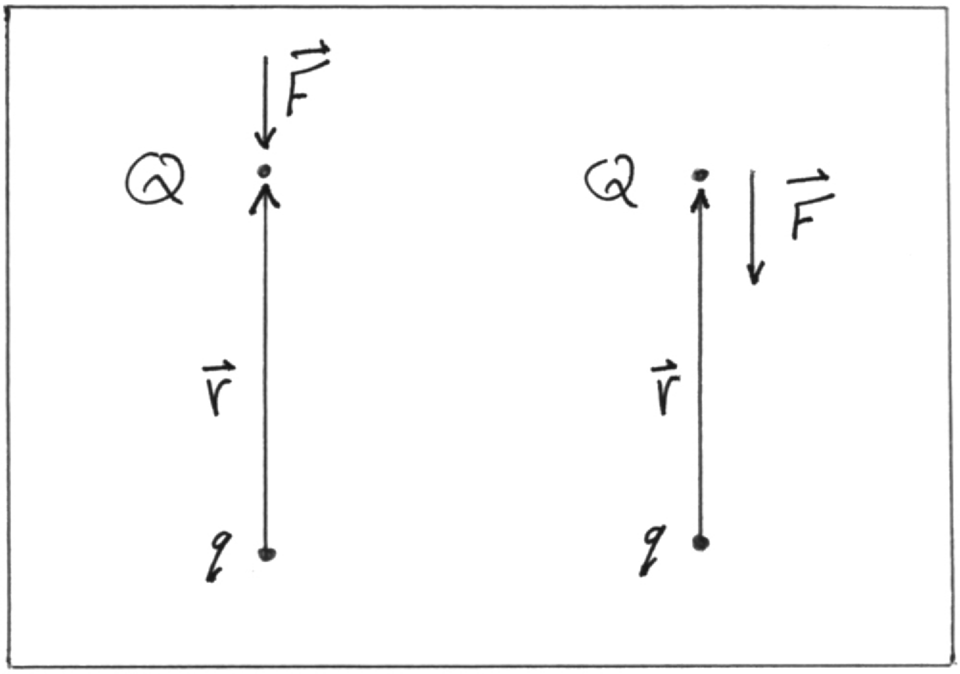
$\vec{E} = 143.8\ \hat{j}\ kN/C$



3- We have : $\vec{E} = \dfrac{\vec{F}}{Q}$

or, $\vec{F} = Q\vec{E} = -18 \times 10^{-6} \times 0.1438 \times 10^{6}\ \hat{j}$

$$\vec{F} = -2.5884\ \hat{j}\ N$$



Gauss' Law in Terms of $\vec{D}$

Integral form

* $\oint_s \vec{E}\cdot\vec{ds} = \dfrac{Q}{\epsilon} \Longrightarrow \oint_s \epsilon\vec{E}\cdot\vec{ds} = Q$

$$\oint_s \vec{D}\cdot\vec{ds} = Q$$

* The integral of electric flux density over a closed surface equals the total charge enclosed by that surface.

Differential Form

* $\vec{\nabla}\cdot\vec{E} = \rho_v/\epsilon \Longrightarrow \epsilon\vec{\nabla}\cdot\vec{E} = \rho_v$

* Since (ϵ) is constant, we may write:

$$\vec{\nabla}\cdot\epsilon\vec{E} = \rho_v \quad , \text{ or, } \quad \vec{\nabla}\cdot\vec{D} = \rho_v$$

* The divergence of the electric flux density at a point equals the volume charge density at that point.

In Summary

Gauss's Law

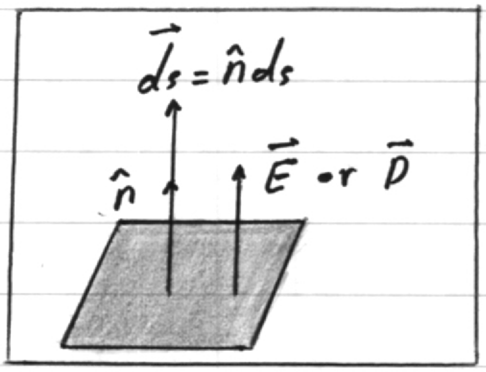
In Terms of	Integral form	Differential form
Electric field Intensity $\{\vec{E}\}$	$\oint_s \vec{E} \cdot \vec{ds} = Q/\epsilon$	$\vec{\nabla} \cdot \vec{E} = \rho_v / \epsilon$
Electric flux Density $\{\vec{D}\}$	$\oint_s \vec{D} \cdot \vec{ds} = Q$	$\vec{\nabla} \cdot \vec{D} = \rho_v$

Symmetry and Gauss' Law

*- Choosing suitable Gauss' surface may simplify the integration in Gauss' law. This depends on the symmetry of the problem.

*- Illustration:

① If $\vec{E}$ or $\vec{D}$ $\perp$ ds :



*- In this case:
$\vec{E} \parallel \vec{ds}$, or, $\vec{D} \parallel \vec{ds}$

*- For example: $\oint_s \vec{D} \cdot \vec{ds} =$

$$= \oint_s D \, ds \cos(0) =$$

$$= \oint_s D \, ds$$

*- If D is constant among the whole surface (S), then:

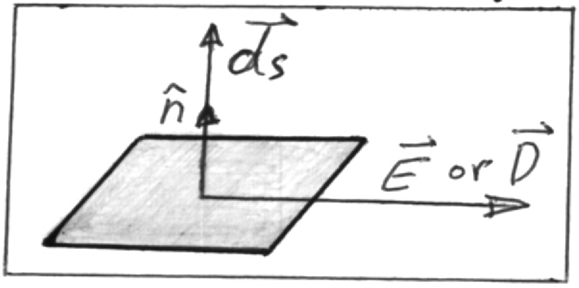
$$\oint_s D \, ds = D \oint_s ds$$

$$= D S$$

$\therefore$ $\int_s \vec{D} \cdot \vec{ds} = D S$ when $\vec{D}$ (or $\vec{E}$) $\perp$ to the surface (S) {for Open or Closed surfaces}.

② If $\vec{E}$ or $\vec{D}$ // ds

*- In this case: $\vec{E} \perp \vec{ds}$ or $\vec{D} \perp \vec{ds}$:



*- $\oint \vec{D} \cdot \vec{ds} = \int D \, ds \cos(90)$

$= 0$

$\therefore \int \vec{D} \cdot \vec{ds} = 0$ when $\vec{D}$ (or $\vec{E}$) // to the surface (S) {for Open or Closed surfaces}.

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*- We shall deal with spherical and cylindrical symmetries :

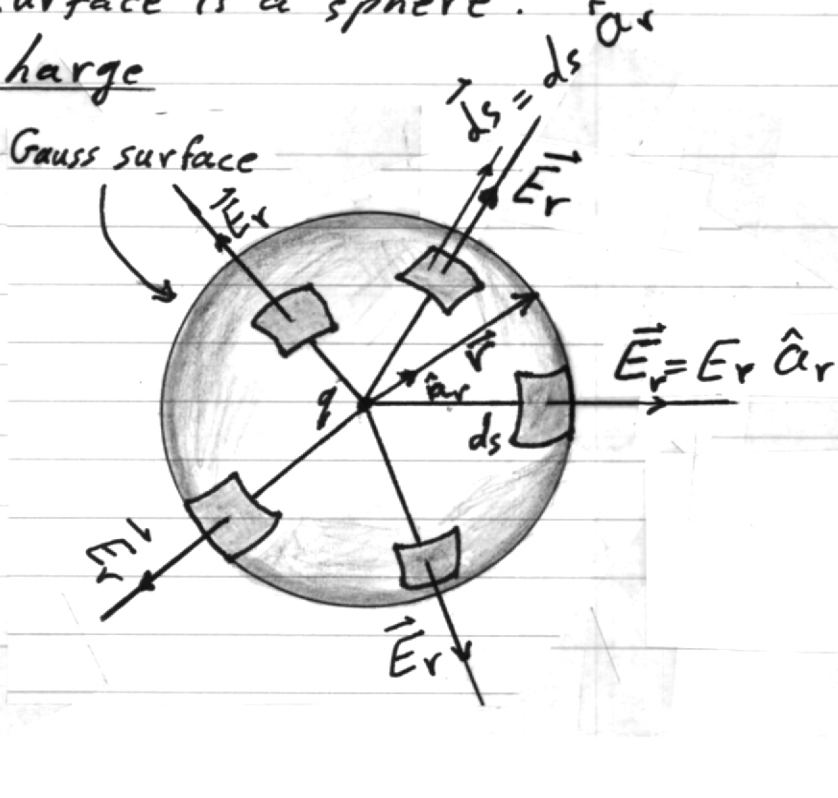
1- <u>Spherical Symmetry:</u> (r, θ, ϕ)

*- The effective component of $\vec{E}$ or $\vec{D}$ is the radial component (r) : $\vec{E} = \vec{E_r} = E_r \hat{a}_r$

or, $\vec{D} = \vec{D_r} = D_r \hat{a}_r$

*- The Gauss surface is a sphere.

<u>a- The point charge</u>



*- In this case :

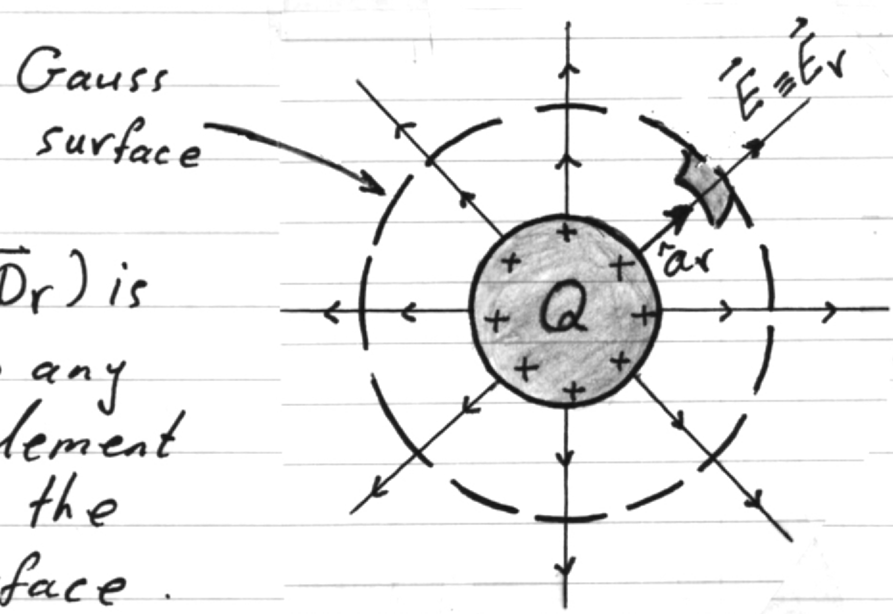
$$\oint_s \vec{E} \cdot \vec{ds} \equiv \oint_s \vec{Er} \cdot \vec{ds} \equiv \oint_s Er\, \hat{a}_r \cdot \hat{a}_r\, ds$$

$$\therefore \oint_s \vec{E} \cdot \vec{ds} \equiv \oint_s Er\, \hat{a}_r \cdot \hat{a}_r\, ds \equiv \oint_s Er\, ds$$

*- For Er is constant over the whole Gauss surface :

$$\oint_s \vec{E} \cdot \vec{ds} = Er \oint_s ds = Er\, S$$

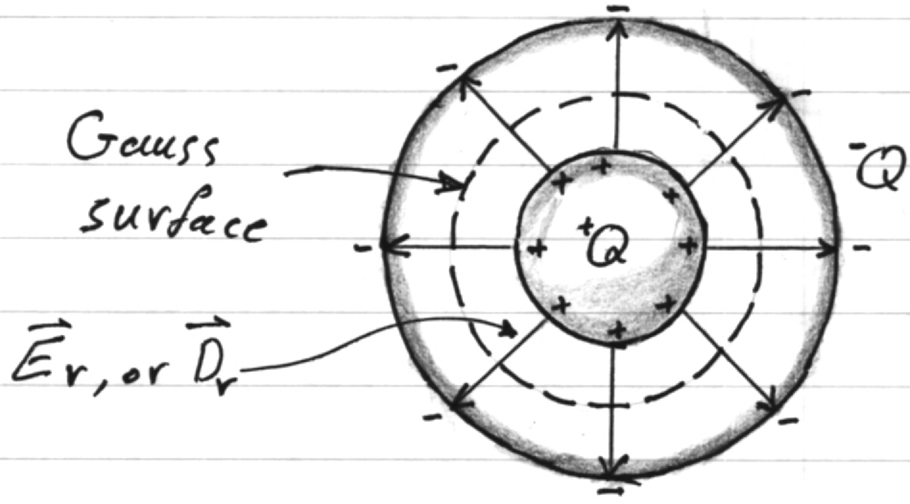
b- A Charged Sphere



*- $\vec{Er}$ (or $\vec{D}_r$) is normal to any surface element (ds) over the Gauss surface.

*- Or, $\vec{D}r \parallel \vec{ds}$.

c- Concentric Charged Spheres



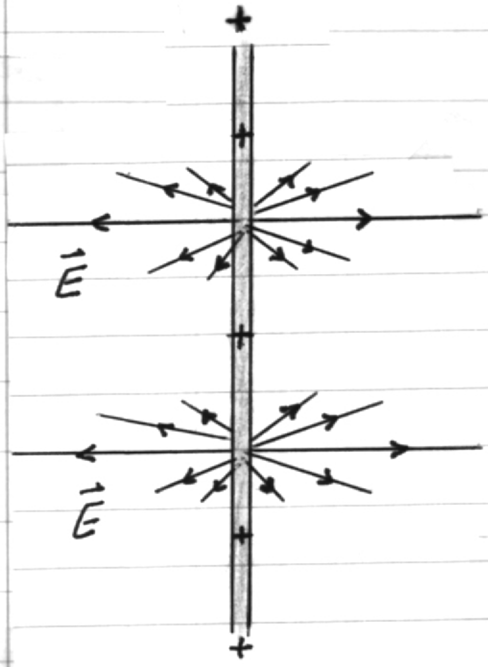
2- Cylindrical Symmetry (ρ, ϕ, z)

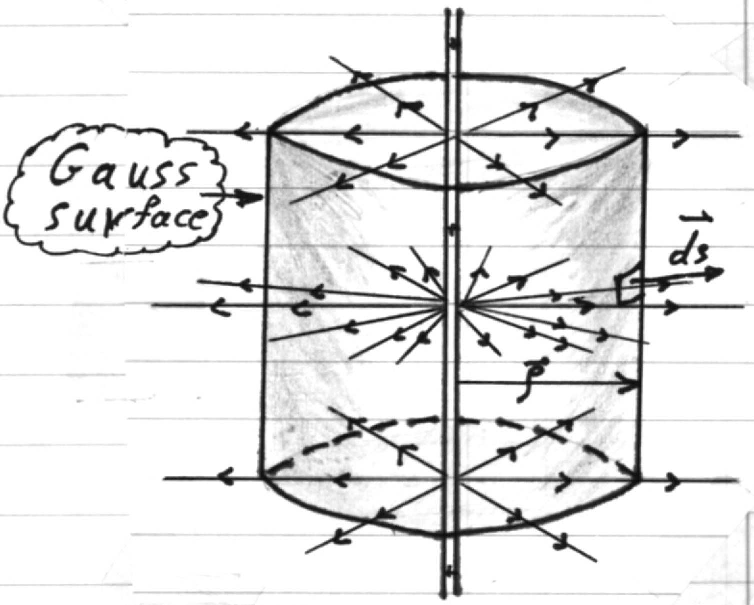
*- The effective component for $\vec{E}$ or $\vec{D}$ is the radial component (along ρ):

$$\vec{E} \equiv \vec{E}_\rho \equiv E_\rho \, \hat{a}_\rho$$
$$\vec{D} \equiv \vec{D}_\rho \equiv D_\rho \, \hat{a}_\rho$$

*- The Gauss surface is a cylinder.

a- Line Charge





*- At the side surface : $\vec{D}_\rho \perp ds$
(or $\vec{D}_\rho \parallel \vec{ds}$) , $\vec{ds} = \hat{a}_\rho ds$

$$\therefore \int_s \vec{D}_\rho \cdot \vec{ds} \equiv \int_s D_\rho \hat{a}_\rho \cdot \hat{a}_\rho ds$$

$$\therefore \int_s \vec{D}_\rho \cdot \vec{ds} \equiv \int_s D_\rho \hat{a}_\rho \cdot \hat{a}_\rho ds$$

$$\equiv \int_s D_\rho ds$$

*- If D_ρ is constant over the whole Gauss surface , then :

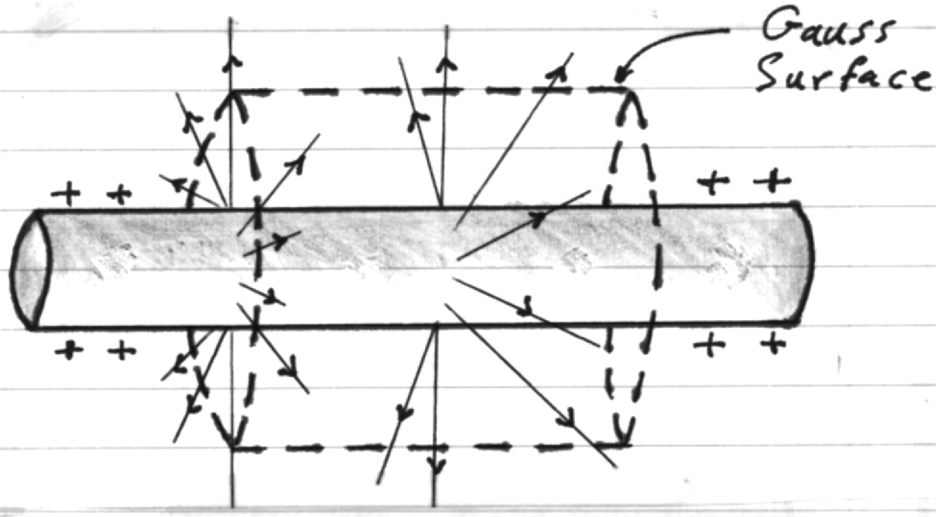
$$\int_s \vec{D}_\rho \cdot \vec{ds} = D_\rho \int_s ds = D_\rho S$$

*- At the top (or bottom) : $\vec{ds} = \hat{a}_z \, ds$

$\vec{D}_\rho \parallel ds$ (or $\vec{D}_\rho \perp \vec{ds}$)

*- In this case $\int_s \vec{D}_\rho \cdot \vec{ds} = 0$

b- A Charged Cylinder



c- Coaxial Charged Cylinders