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- EM6 -

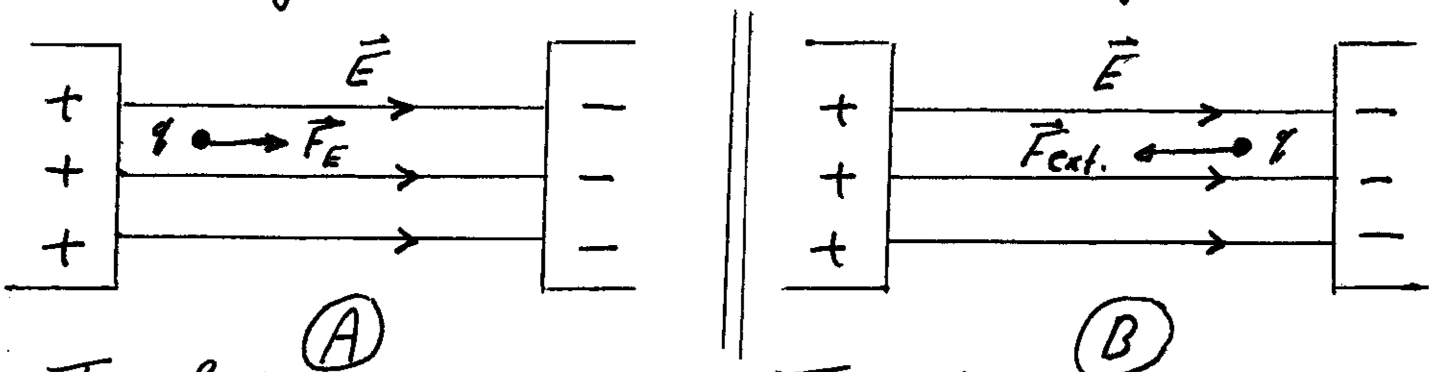
نظريه المجالات
قسم هندسة الاتصالات
المرحلة الثالثه

Energy and Potential

*- The force exerted on an electric charge placed in an electric field could be due to:

1- The field ($\vec{F}_E = q\vec{E}$)
2- External force ($\vec{F}_{ext.}$)
3- Both.

*- Taking, for example, a positive charge (q):



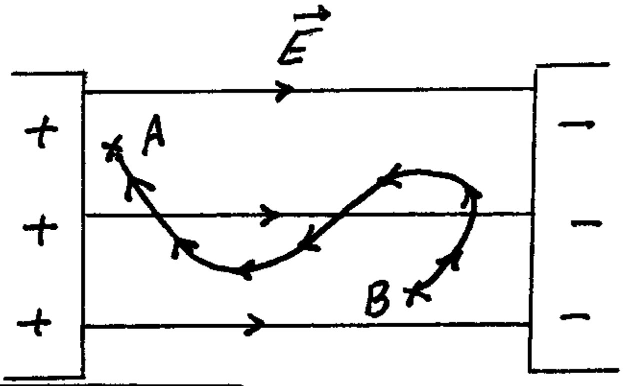
The field force moves the charge to the right. | The external force moves the charge to the left.

*- In both cases, (A) and (B), since there is a force and displacement, then there is a Work (energy), {units: Joul (J)}.

*- Case (A): The work is being expended by the field (-Ve).

*- Case (B): The work is being expended by the external force (+Ve).

* In general, the work expended in moving a charge (q) from an initial location (B) (of lower potential) to a final location (A) (of higher potential) is given as:



$$\boxed{W = -q \int_B^A \vec{E} \cdot \vec{dl}}$$

$\vec{dl}$: a displacement element along the path $B \to A$.

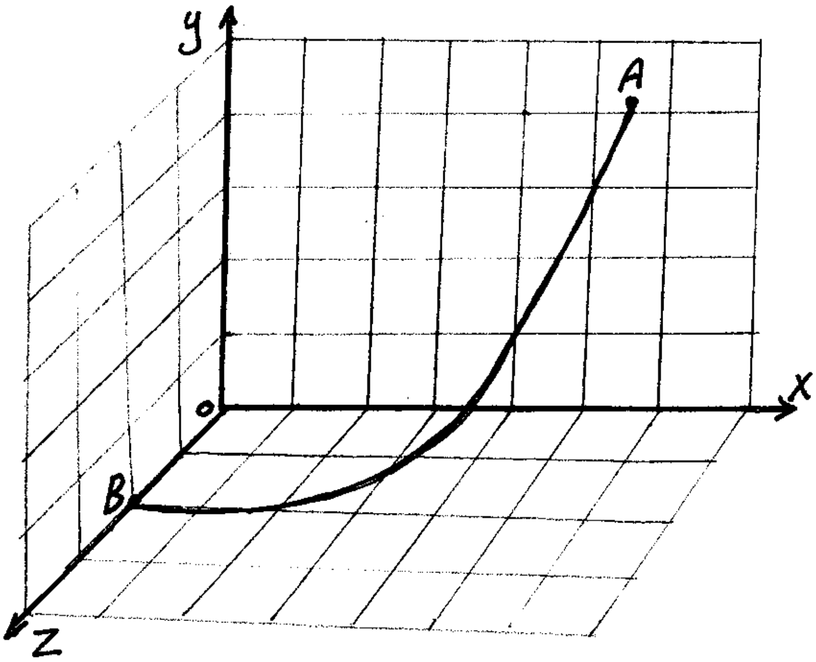
Ex. 1: For the E-field: $\vec{E} = y\,\hat{i} + x\,\hat{j} + 2\,\hat{k}$, determine the work expended in carrying a 2C charge from $B(0,0,1)$m to $A(2,4,1)$m along the path: $y = x^2$, $z = 1$.

Sol.

We have:

$y = x^2$

or, $x = y^{1/2}$



Let the path is $\vec{L}$:

$$\vec{L} = \hat{i}\,x + \hat{j}\,y + \hat{k}\,z$$

$$\therefore \vec{dl} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz$$

$$W = -q\int_B^A \vec{E}\cdot\vec{dl}$$

$$W = -q\int_B^A (y\hat{i} + x\hat{j} + 2\hat{k})\cdot(\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$= -q\int_B^A (y\,dx + x\,dy + 2\,dz)$$

$$= -q\left[\int_0^2 y\,dx + \int_0^4 x\,dy + 2\int_1^1 dz\right]$$

$$= -q\left[\int_0^2 x^2\,dx + \int_0^4 y^{1/2}\,dy + 2\int_1^1 dz\right]$$

$$= -q\left\{\left[\frac{x^3}{3}\right]_0^2 + \left[\frac{y^{3/2}}{3/2}\right]_0^4 + 2\left[z\right]_1^1\right\}$$

$$= -q\left\{\frac{1}{3}(2^3-0) + \frac{2}{3}(4^{1.5}-0) + 2(1-1)\right\}$$

$$= -q\left\{\frac{8}{3} + \frac{2}{3}\times 8 + 2\times(0)\right\}$$

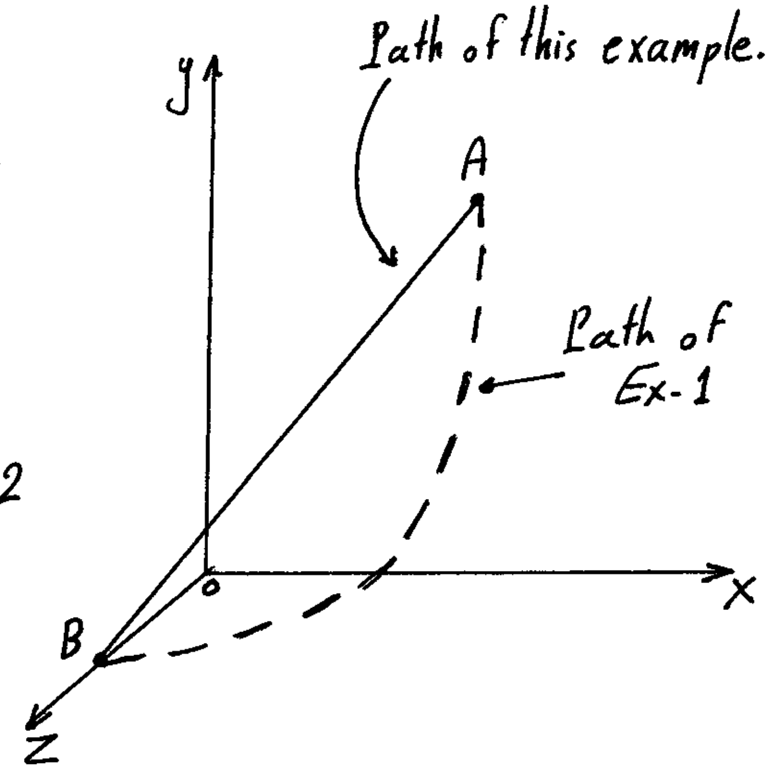
$$= -2\left\{\frac{8}{3} + \frac{16}{3}\right\} = -2\times\frac{24}{3}$$

$$= -2\times 8 = -16$$

$$\therefore\ W = -16\ J.$$

<u>ملاحظة</u>: الأشارة السالبه تدل على أن المجال الكهربائي هو الذي يقوم بأنجاز هذا الشغل .

Ex. 2 : Repeat Ex.1 using the straight-line path : $y=2X$, $Z=1$.



Sol.

We have :

$q = 2\,C$

$\vec{E} = y\hat{\imath} + x\hat{\jmath} + 2\hat{k}$

$y = 2X$, or, $X = y/2$

$\vec{dl} = \hat{\imath}\,dx + \hat{\jmath}\,dy + \hat{k}\,dz$

$$W = -q\int_B^A \vec{E}\cdot\vec{dl}$$

$$W = -q\int_B^A (y\hat{\imath}+x\hat{\jmath}+2\hat{k})\cdot(\hat{\imath}\,dx+\hat{\jmath}\,dy+\hat{k}\,dz)$$

$$= -q\int_B^A (y\,dx + x\,dy + 2\,dz)$$

$$= -q\left\{\int_0^2 y\,dx + \int_0^4 x\,dy + 2\int_1^1 dz\right\}$$

$$= -q\left\{\int_0^2 2X\,dx + \int_0^4 \frac{y}{2}\,dy + 2\int_1^1 dz\right\}$$

$$= -q\left\{2\int_0^2 x\,dx + \frac{1}{2}\int_0^4 y\,dy + 2\int_1^1 dz\right\}$$

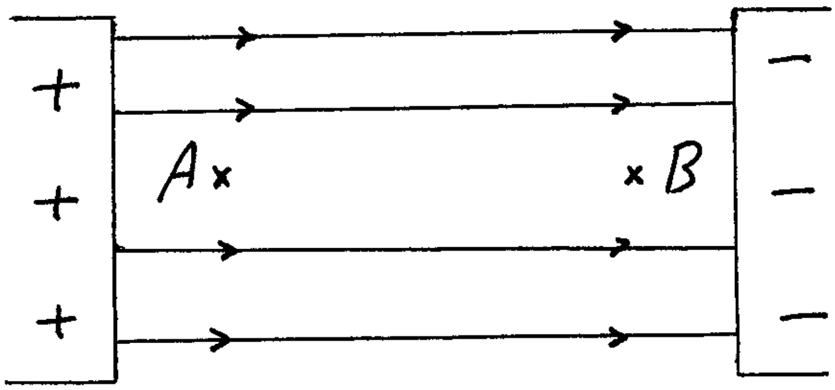
$$= -q\left\{2\left[\frac{X^2}{2}\right]_0^2 + \frac{1}{2}\left[\frac{y^2}{2}\right]_0^4 + 2\left[Z\right]_1^1\right\}$$

$$W = -q\left\{[x^2]_0^2 + \frac{1}{4}[y^2]_0^4 + 2[z]_1^1\right\}$$
$$= -q\left\{(4-0) + \frac{1}{4}(16-0) + 2(1-1)\right\}$$
$$= -2\left\{4 + \frac{16}{4} + 2\times 0\right\} = -2\,(4+4)$$
$$= -2\times 8 = -16$$

$\therefore W = -16\ \text{J}$ (The same result of Ex.1).

ملاحظة: تطابق الناتج في هذين المثالين يدل على أن مقدار الشغل لا يعتمد على **شكل** المسار (path) المحدد في المسألة.

Potential and Potential Difference



*- Since there is an amount of work (energy) expended in moving an electric charge between points A and B, therefore there is a potential difference between them V_{AB} :

$$V_{AB} = V_A - V_B$$

*- Potential difference is defined as the work expended per unit charge.

$$V_{AB} = \frac{W}{q} \quad ; \quad \underline{\text{Units}} : \frac{J}{C} \equiv \text{Volt (V)}.$$

$$\boxed{V_{AB} = -\int_B^A \vec{E} \cdot \vec{dl} = V_A - V_B}$$

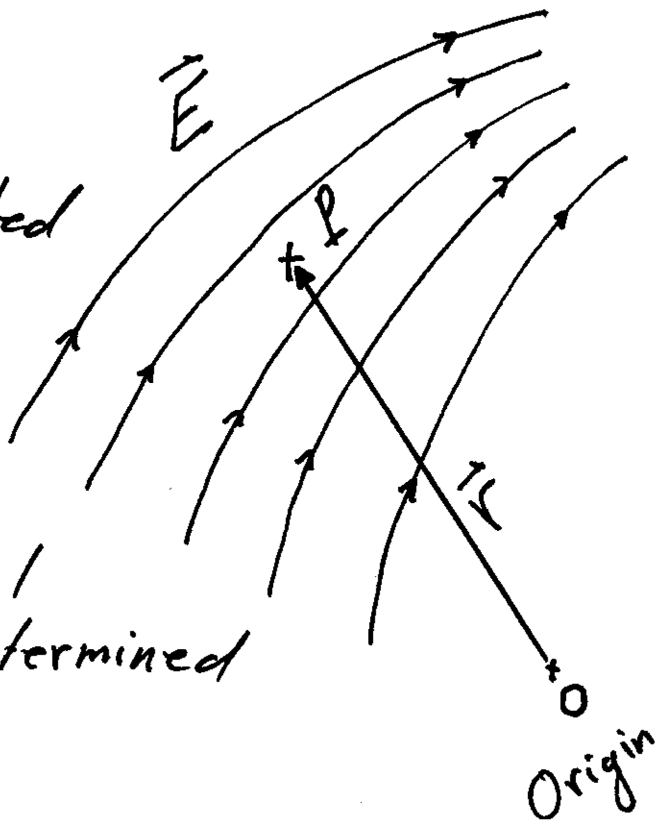
Try: Use this equation to find V_{AB} in Ex. 1 \ P. 2.
Which point is of the higher potential?

The Potential of a Single Point

*- The electric potential of a point is defined as the work expended in moving a unit charge from a reference location (where the potential is zero) to that point.

{Usually, the reference location is considered at infinity}.

*- Illustration:



→ A point (P) is located in an electric field.

→ The location of (P) is ($\vec{r}$).

→ The electric potential at point (P) is determined as:

$$V = -\int_{\infty}^{r} \vec{E} \cdot \vec{dl}$$

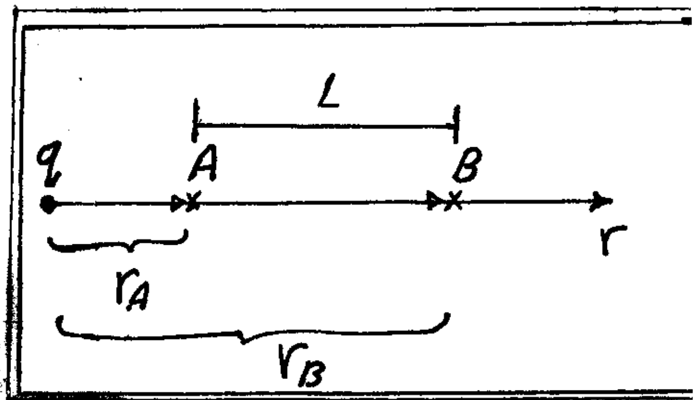
or, $\boxed{V = -\int \vec{E} \cdot \vec{dl}}$

{Units of the electric potential is (V)}.

For a point charge

*- Potential difference between two points A and B placed in a field of a point charge (q) is given as:

$$V_{AB} = -\int_B^A \vec{E} \cdot \vec{dl} = V_A - V_B$$



*- The displacement between B and A is along the r-coordinate

$(L \to r)$, $(\vec{l} \to \vec{r})$, $(\vec{dl} \to \vec{dr})$

$$\therefore V_{AB} = -\int_B^A \vec{E} \cdot \vec{dr}$$

for a point charge, we have: $\vec{E} = k \dfrac{q}{r^2} \hat{a}_r$

we have: $\vec{dr} = \hat{a}_r \, dr$

$$V_{AB} = -\int_B^A k \frac{q}{r^2} \hat{a}_r \cdot \hat{a}_r \, dr = -\int_B^A k \frac{q}{r^2} \, dr$$

$$V_{AB} = -kq \int_{r_B}^{r_A} \frac{dr}{r^2} = -kq \left[-r^{-1} \right]_{r_B}^{r_A}$$

$$V_{AB} = kq \left[r_A^{-1} - r_B^{-1} \right] = kq \left[\frac{1}{r_A} - \frac{1}{r_B} \right]$$

$$V_{AB} = kq \left[\frac{1}{r_A} - \frac{1}{r_B} \right] \quad \text{——} \quad \alpha$$

or:

$$V_{AB} = kq\left[\frac{r_B - r_A}{r_A r_B}\right] \quad ; \; V_{AB} = \frac{q}{4\pi\epsilon}\left[\frac{r_B - r_A}{r_A r_B}\right]$$

*- V_{AB} is the potential difference between the two points A and B.

*- To get the potential of a single point, we must compare it's potential with that of a reference position at which the potential is zero. {Usually reference position is taken as ∞}.

$\Rightarrow$ To get the potential of point A from eq. ⓐ we let point B ~~to~~ be at infinity.

*- From eq. ⓐ, with $r_B \to \infty$, we have:

$$V_A = kq\left[\frac{1}{r_A} - \frac{1}{\infty}\right] \; ; \; V_A = k\frac{q}{r_A}$$

*- <u>In general</u>, the electric potential due to a point charge at a point of a distance (r) from this point charge, with ∞ as a reference position is:

$$\boxed{V = k\frac{q}{r} = \frac{q}{4\pi\epsilon r}} \qquad (V) \quad \text{unit}$$

*- To express the potential without selecting a specific reference position, we may write:

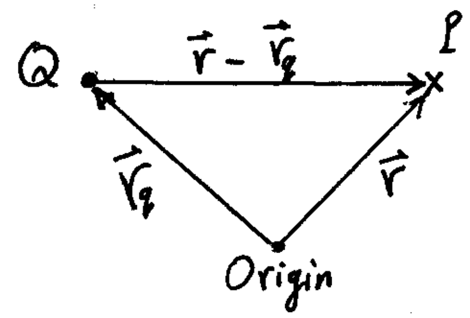
$$\boxed{V = k\frac{q}{r} + C} \quad ; \quad C\text{ - Value is selected}$$

so that $V = 0$ at any desired value of r.

*- Potential Field of a System of Charges :

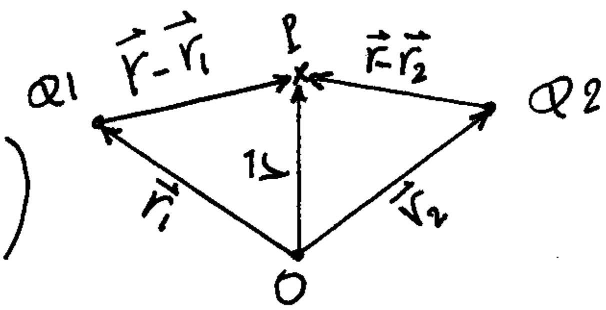
A point charge :

$$V = k \frac{Q}{|\vec{r} - \vec{r}_q|}$$



Two point charges :

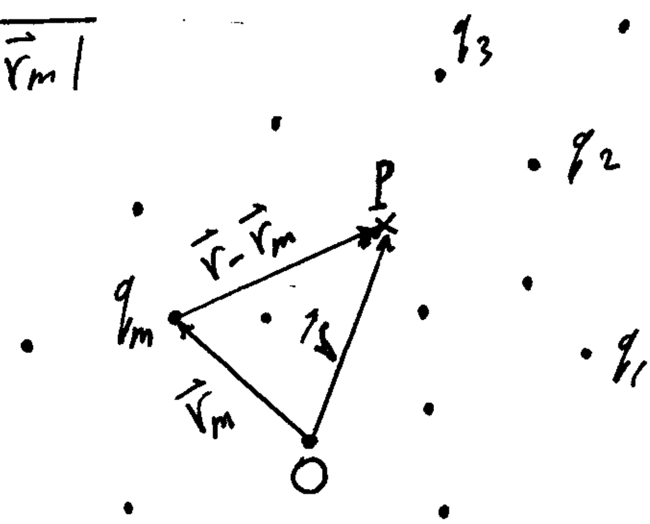
$$V = k \left(\frac{Q1}{|\vec{r} - \vec{r}_1|} + \frac{Q2}{|\vec{r} - \vec{r}_2|} \right)$$



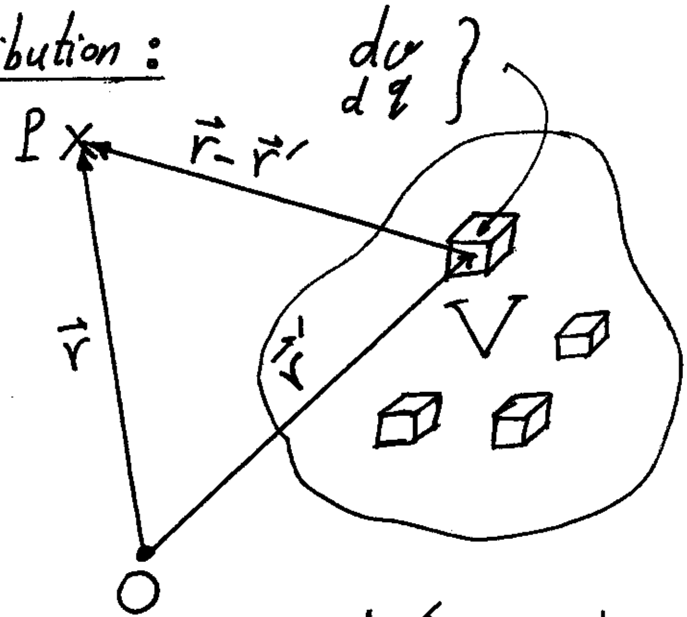
Any number of point charges :

For any No. of point charges (n),

$$V = k \sum_{m=1}^{n} \frac{q_m}{|\vec{r} - \vec{r}_m|}$$



Volume charge distribution :



$$V = k \int_V \frac{dq}{|\vec{r}-\vec{r}\,'|}$$

$\rho_v = dq / dv$

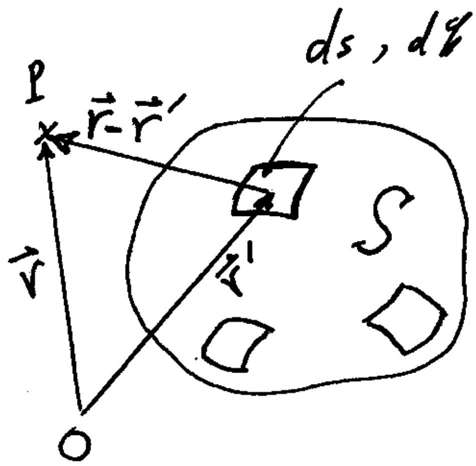
or, $dq = \rho_v \, dv$

$$\therefore \; V = k \int_V \frac{\rho_v \, dv}{|\vec{r}-\vec{r}\,'|}$$

dq: مقدار الشحنه الكهربائية

dv داخل العنصر الحجمي

Surface charge distribution:



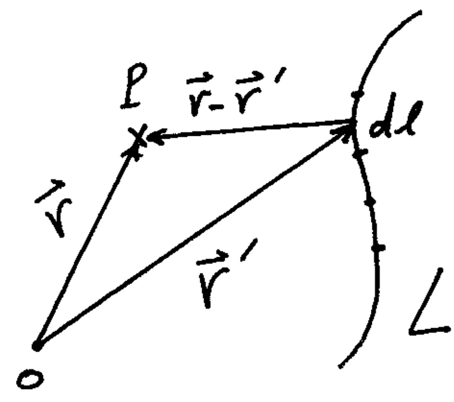
$$V = k \int_S \frac{dq}{|\vec{r}-\vec{r}\,'|}$$

$\rho_s = dq / ds$

or, $dq = \rho_s \, ds$

$$\therefore \; V = k \int_S \frac{\rho_s \, ds}{|\vec{r}-\vec{r}\,'|}$$

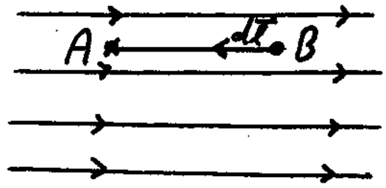
Line charge distribution :



$$V = k \int_L \frac{\rho_L \, dl}{|\vec{r}-\vec{r}\,'|}$$

Conservative Property

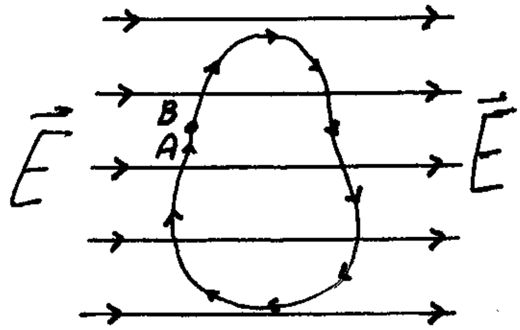
- We have:



$$V_{AB} = -\int_B^A \vec{E}\cdot\vec{dl} = V_A - V_B$$

*- If the path from B to A is a closed path:

$A \equiv B$, and :



$$\oint \vec{E}\cdot\vec{dl} = V_B - V_A = 0$$

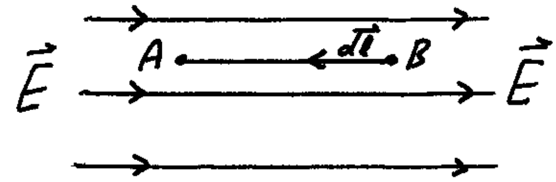
$$\therefore \oint \vec{E}\cdot\vec{dl} = 0$$

*- The field that has a zero closed line integral is a conservative field.
{In this case the $\vec{E}$-field is a conservative field}.

*- Electric Potential at a Point in Terms of $\vec{E}$:

*- We have : $V_{AB} = -\int_{B}^{A} \vec{E} \cdot \vec{dl} = V_A - V_B$

*- We shall consider (∞) as a reference position at which the potential is zero.



*- Let (B) is at $\infty \Longrightarrow V_B = 0$

$$\therefore \quad -\int_{\infty}^{A} \vec{E} \cdot \vec{dl} = V_A$$

{ when $V = 0 \Longrightarrow E = 0$ }

*- The above relation may be written in a general form as :

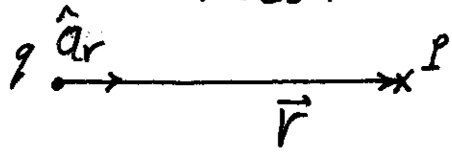
$$\boxed{V = -\int \vec{E} \cdot \vec{dl}} \qquad \text{—— (b)}$$

*- This relation gives the potential of a point in an electric field.

Potential Gradient — الانحدار الجهدي

*- For a point charge we have $V = \dfrac{q}{4\pi\epsilon_\circ r}$

*- Taking the gradient of both sides :



$$\vec{\nabla} V = \vec{\nabla} \frac{q}{4\pi\epsilon_\circ r} \quad \text{—— (c)}$$

*- The only variable in eq. (c) is the distance (r) {from charge q to point P}

$\therefore \quad \vec{\nabla} = \hat{a}_r \dfrac{d}{dr}$, and (c) becomes :

$$\vec{\nabla} V = \frac{d}{dr}\left(\frac{q}{4\pi\epsilon_\circ r}\right) \hat{a}_r$$

$$\vec{\nabla} V = \frac{q}{4\pi\epsilon_\circ} \hat{a}_r \frac{d}{dr}\left(r^{-1}\right)$$

$$\vec{\nabla} V = -\frac{q\,\hat{a}_r}{4\pi\epsilon_\circ} r^{-2} = \frac{-q}{4\pi\epsilon_\circ r^2}\hat{a}_r = -E\,\hat{a}_r = -\vec{E}$$

$\therefore \quad \vec{\nabla} V = -\vec{E}$

Or, $$\boxed{\vec{E} = -\vec{\nabla} V} \quad \text{—— (d)}$$

Other units for (E) : $E : \dfrac{V}{m}$

*- العلاقة (d) هي عكس العلاقة السابقة (b) :

$$\boxed{V = -\int \vec{E} \cdot \vec{dl}}$$

Notes :

*- Forms of differential path (line element) in coordinate systems :

$$\vec{dl} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz \qquad [\text{Cartesian}].$$

$$\vec{dl} = \hat{a}_\rho\,d\rho + \hat{a}_\phi\,\rho\,d\phi + \hat{a}_z\,dz \qquad [\text{Cylindrical}]\ (\hat{a}_z \equiv \hat{k})$$

$$\vec{dl} = \hat{a}_r\,dr + \hat{a}_\theta\,r\,d\theta + \hat{a}_\phi\,\sin\theta\,d\phi \qquad [\text{Spherical}]$$

*- $\vec{dl}$ (أو $\vec{dL}$) هو عنصر تفاضلي (أي صغير جداً جداً) على المسار الذي يصل بين نقطتين ، أو قد يكون مسار مغلق .

<u>Hw.</u>: Calculate the work done in moving a 4 C charge from B(1,0,0)m to A(0,2,0)m along the path $y = 2 - 2X$, $Z = 0$ in the field:

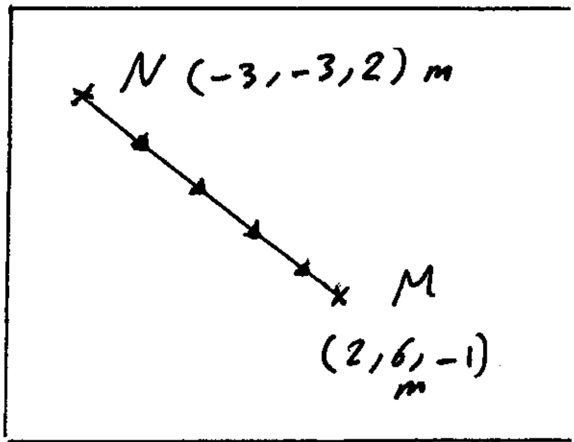
(a): $\vec{E} = 5\,\hat{a}_x$ V/m.
(b): $\vec{E} = 5X\,\hat{a}_x$ V/m.
(c): $\vec{E} = 5X\,\hat{a}_x + 5y\,\hat{a}_y$ V/m

<u>Ans.</u>: <u>(a)</u>: 20 J , <u>(b)</u>: 10 J , <u>(c)</u>: -30 J .

Ex. 3

An electric field is expressed in cartesian coordinates by $\vec{E} = 6x^2\hat{i} + 6y\hat{j} + 4\hat{k}$ (V/m).

Find: (a): V_{MN}, (b): V_M, (c): V_N, (d): V_S, (e): V_T, where: $M(2,6,-1)$ m, $N(-3,-3,2)$ m, $T(4,-2,-35)$ m, and $S(1,2,-4)$ m.



Sol.: (a):

$$V_{MN} = V_M - V_N$$

$$V_{MN} = -\int_N^M \vec{E}\cdot d\vec{l}$$

$$\vec{E} = 6x^2\hat{i} + 6y\hat{j} + 4\hat{k}$$

$$d\vec{l} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz$$

$$V_{MN} = -\int_N^M (6x^2\hat{i} + 6y\hat{j} + 4\hat{k})\cdot(\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$V_{MN} = \int_M^N (6x^2dx + 6y\,dy + 4\,dz)$$

$$= 6\int_2^{-3} x^2dx + 6\int_6^{-3} y\,dy + 4\int_{-1}^{2} dz$$

$$= 6\left[\frac{x^3}{3}\right]_2^{-3} + 6\left[\frac{y^2}{2}\right]_6^{-3} + 4\left[z\right]_{-1}^{2}$$

$$V_{MN} = \frac{6}{3}\left[x^3\right]_2^{-3} + \frac{6}{2}\left[y^2\right]_6^{-3} + 4\left[z\right]_{-1}^{2}$$

$$= 2\left[(-3)^3 - 2^3\right] + 3\left[(-3)^2 - 6^2\right] + 4\left[2-(-1)\right]$$

$$= 2\left[-27-8\right] + 3\left[9-36\right] + 4\left[3\right]$$

$$= 2(-35) + 3(-27) + 12$$

$$= -70 - 81 + 12 = -139$$

$$V_{MN} = -139 \text{ V}.$$

<u>(b):</u> $V = -\int \vec{E} \cdot \vec{dl}$

$$V = -\int (6x^2\hat{i} + 6y\hat{j} + 4\hat{k}) \cdot (\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$V = -(2x^3 + 3y^2 + 4z)$$

$$V_M = -(2 \times 2^3 + 3 \times 6^2 - 4) = -(16 + 108 - 4)$$

$$V_M = -120 \text{ V}.$$

<u>(c):</u> $V_{MN} = V_M - V_N$

$$V_N = V_M - V_{MN}$$

$$V_N = -120 + 139 = 19 \text{ V}.$$

<u>(d)</u>: Ans.: $V_S = 2$ V.

<u>(e)</u>: Ans.: $V_T = 0$ V.

Ex. 4:

A 15 nC point charge at the origin. Calculate the potential at point P(−2, 3, −1) m, if: (a) V = 0 at infinity, (b) V = 0 at S(6, 5, 4) m, (c) V = 5 V at T(2, 0, 4) m.

Sol. (a):

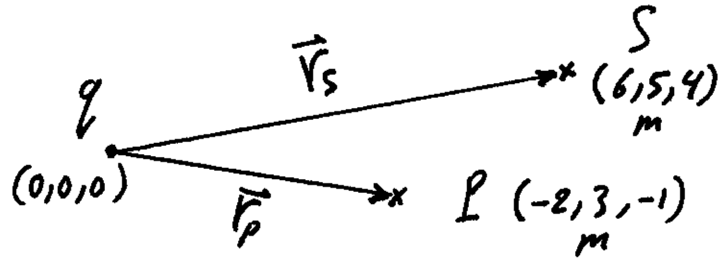
$$V_P = k\,\frac{q}{r_P}$$

$$\vec{r}_P = -2\hat{i} + 3\hat{j} - \hat{k} \Rightarrow r_P = \sqrt{4+9+1} = \sqrt{14}$$
$$r_P = 3.7416 \text{ m}$$

$$V_P = 9\times10^{9}\,\frac{15\times10^{-9}}{3.7416} = 36.0808 \text{ V}$$

$$V_P = 36.0808 \text{ V}.$$

(b):



$$V_{PS} = V_P - V_S$$

$$V_P = V_{PS} + V_S = V_{PS} + 0$$

$$\therefore V_P = V_{PS}$$

Using eq. ⓐ (P.8):

$$V_{PS} = kq\left[\frac{1}{r_P} - \frac{1}{r_S}\right]$$

$$\vec{r}_S = 6\hat{i} + 5\hat{j} + 4\hat{k} \Longrightarrow r_S = \sqrt{36+25+16} = \sqrt{77}$$

$r_S = 8.7749$ m.

We have: $r_P = 3.7416$ m

$$V_{PS} = 9\times10^{9}\times15\times10^{-9}\left[\frac{1}{3.7416} - \frac{1}{8.7749}\right]$$

$$V_{PS} = 135\left[0.2672 - 0.1139\right] = 135(0.1533)$$

$$V_{PS} = 20.6955 \text{ v} \Longrightarrow \therefore V_P = 20.6955 \text{ v.}$$

(c): H.W.

Hints: $V_{PT} = V_P - V_T \Rightarrow V_P = V_{PT} + V_T$

$$V_P = V_{PT} + 5$$

$$V_{PT} = kq\left[\frac{1}{r_P} - \frac{1}{r_T}\right]$$

Ans.: $V_P = 10.886$ v.

Ex. 5 :

Point $P(-4,3,6)$m in a potential field given as : $V = 2X^2y - 5Z$.
At point (P), find the followings :
(a) the potential (V_P), (b) electric field intensity $(\vec{E}_P)$, (c) the direction of $(\vec{E}_P)$, (d) electric flux density $\vec{D}_P$ (the medium is free space), (e) the volume charge density $(\rho_v)_P$.

Sol.:

(a): $V_P = 2(-4)^2(3) - 5(6) = 66$ v

$V_P = 66$ v.

(b): $\vec{E} = -\vec{\nabla} V$

$$\vec{E} = -\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(2x^2y - 5z)$$

$$\vec{E} = -4xy\,\hat{i} - 2x^2\,\hat{j} + 5\,\hat{k} \quad (\mathrm{v/m}).$$

$$\vec{E}_P = -4(-4)(3)\hat{i} - 2(-4)^2\,\hat{j} + 5\hat{k}$$

$$\vec{E}_P = 48\,\hat{i} - 32\,\hat{j} + 5\hat{k} \quad (\mathrm{v/m})$$

(c): The direction of $\vec{E}_P$ is the direction of it's unit vector $(\hat{a}_{EP})$:

$$\vec{E}_P = E_P\ \hat{a}_{EP} \Rightarrow \hat{a}_{EP} = \frac{\vec{E}_P}{E_P}$$

$$E_p = \sqrt{48^2 + (-32)^2 + 5^2} = \sqrt{3353} = 57.9050 \frac{V}{m}.$$

$$\hat{a}_{E\ell} = \frac{1}{57.905}(48\hat{i} - 32\hat{j} + 5\hat{k})$$

$$\hat{a}_{EP} = 0.8289\hat{i} - 0.5526\hat{j} + 0.0863\hat{k}$$

(d): $\vec{D}_P = \epsilon_0 \vec{E}_\ell = 8.854\times10^{-12}(48\hat{i} - 32\hat{j} + 5\hat{k})$

$$\vec{D}_\ell = 424.992\times10^{-12}\hat{i} - 283.328\times10^{-12}\hat{j} + 44.27\times10^{-12}\hat{k}$$

$$\vec{D}_P = (424.992\hat{i} - 283.328\hat{j} + 44.27\hat{k})\times10^{-12}\, C/m^2$$

$$\boxed{\text{or, } \vec{D}_P = 424.992\hat{i} - 283.328\hat{j} + 44.27\hat{k} \quad pC/m^2.}$$

(e): $\rho_v = \vec{\nabla}\cdot\vec{D}$ (Differential Gauss' Law).

$$\vec{D} = \epsilon_0\vec{E} = \epsilon_0(-4xy\hat{i} - 2x^2\hat{j} + 5\hat{k})$$

$$\rho_v = \vec{\nabla}\cdot\vec{D} = \left(\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\epsilon_0(-4xy\hat{i} - 2x^2\hat{j} + 5\hat{k}\right)$$

$$= \epsilon_0\left[\left(\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(-4xy\hat{i} - 2x^2\hat{j} + 5\hat{k}\right)\right]$$

$$\rho_v = \epsilon_0(-4y) = -4\epsilon_0 y$$

$$\rho_v = -4\epsilon_0 y$$

At point P :

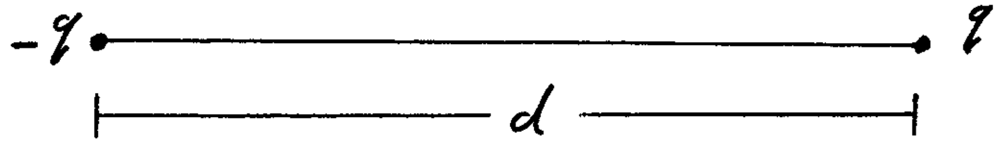
$$(\rho_v)_P = -4 \times 8.854 \times 10^{-12} \times 3$$

$$(\rho_v)_P = -106.248 \times 10^{-12} \quad C/m^3$$

$$\boxed{\text{or, } (\rho_v)_P = -106.248 \quad pC/m^3}$$

The Electric Dipole

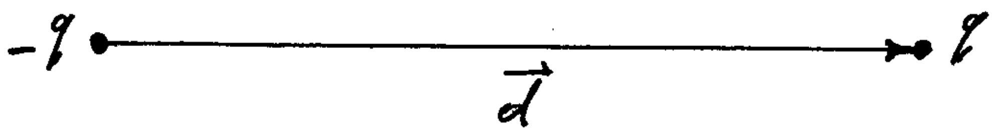
*- Two equal and opposite charges separated by a distance that is large compared to the charges dimensions.



*- The electric dipole is characterized by a moment (p) defined as:

The electric dipole moment: $p = q\,d$.

*- <u>Introducing directions:</u>



$$\vec{p} = q\,\vec{d}$$

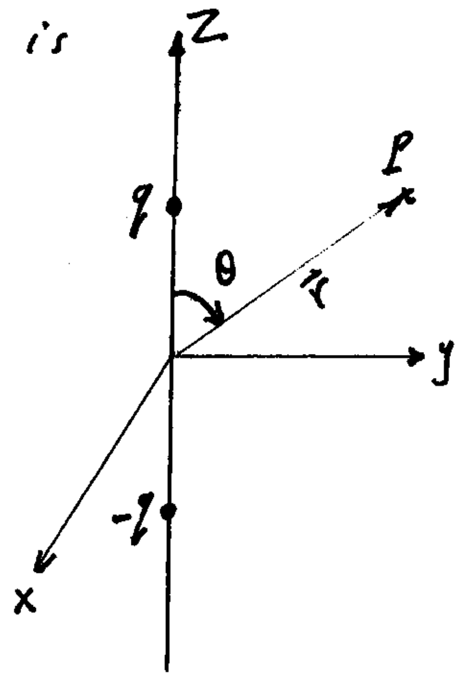
*- The direction of the dipole moment ($\vec{p}$) is from the negative charge toward the positive charge.

*- The electric dipole has effects upone it's surroundings.

*- We may determine the electric potential and field intensity ($\vec{E}$) at any external point.

*- We will describe the location of the external point either by the orthogonal coordinates (x, y, z), or polar coordinates (r, θ).

*- In this illustration, the dipole is along the z-axis.



*- The electric potential at point (P) is given as:

a: $$\boxed{V = \frac{p \cos\theta}{4\pi \epsilon r^2}}$$

b: $$\boxed{V = \frac{\vec{p} \cdot \vec{r}}{4\pi \epsilon r^3}}$$

*- Electric field intensity of a dipole :

*- We have : $\vec{E} = -\vec{\nabla} V$

*- In spherical coordinates :

$$\vec{\nabla} = \hat{a}_r \frac{\partial}{\partial r} + \hat{a}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{a}_\phi \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi}$$

$$\therefore \vec{E} = -\vec{\nabla} V = -\left(\hat{a}_r \frac{\partial V}{\partial r} + \hat{a}_\theta \frac{1}{r}\frac{\partial V}{\partial \theta} + \hat{a}_\phi \frac{1}{r \sin\theta}\frac{\partial V}{\partial \phi}\right) \quad \text{(a)}$$

*- From: $V = \dfrac{P \cos\theta}{4\pi\epsilon_0 r^2} \Longrightarrow$

$$\frac{\partial V}{\partial r} = \frac{P\cos\theta}{4\pi\epsilon_0}\frac{d}{dr}(r^{-2}) = \frac{-2P\cos\theta}{4\pi\epsilon_0} r^{-3}$$

$$\boxed{\frac{\partial V}{\partial r} = \frac{-2P\cos\theta}{4\pi\epsilon_0 r^3}}$$

$$\frac{\partial V}{\partial \theta} = \frac{P}{4\pi\epsilon_0 r^2}\frac{d}{d\theta}(\cos\theta) = \frac{-P}{4\pi\epsilon_0 r^2}\sin\theta$$

$$\boxed{\frac{\partial V}{\partial \theta} = \frac{-P\sin\theta}{4\pi\epsilon_0 r^2}}$$

$$\boxed{\frac{\partial V}{\partial \phi} = 0}$$

, Because V is not a function of ϕ (V does not depend on ϕ).

*- Substituting in (a) :

$$\vec{E} = -\left(\frac{-2P\cos\theta}{4\pi\epsilon_0 r^3}\hat{a}_r - \frac{1}{r}\frac{P\sin\theta}{4\pi\epsilon_0 r^2}\hat{a}_\theta + 0\right)$$

$$\boxed{\vec{E} = \frac{P}{4\pi\epsilon_0 r^3}\left(2\cos\theta\,\hat{a}_r + \sin\theta\,\hat{a}_\theta\right)}$$

Ex. 1: A dipole of moment $\vec{P} = 6\hat{k}$ nC.m is located at the origin in free space.

(a): Find V at $P(r = 4\,m,\ \theta = 20^\circ)$.

(b): Find $\vec{E}$ at P.

Sol.:

(a): We have: $V = \dfrac{p\cos\theta}{4\pi\epsilon_\circ r^2}$

$$p = |\vec{P}| = \sqrt{6^2} = 6 \text{ nC.m} = 6\times10^{-9} \text{ C.m}.$$

$$\therefore V = \frac{6\times10^{-9}\times\cos(20)}{4\times3.14\times8.854\times10^{-12}\times4^2} = \frac{5.6381\times10^{-9}}{1780.202\times10^{-12}}$$

$$V = 3.1671 \text{ V}.$$

(b): $$\vec{E} = \frac{p}{4\pi\epsilon_\circ r^3}\left(2\cos\theta\,\hat{a}_r + \sin\theta\,\hat{a}_\theta\right)$$

$$\vec{E} = \frac{6\times10^{-9}}{4\times3.14\times8.854\times10^{-12}\times4^3}\left(2\cos(20)\,\hat{a}_r + \sin(20)\,\hat{a}_\theta\right)$$

$$\vec{E} = \frac{6\times10^{3}}{7117.1992}\left(2(0.9396)\,\hat{a}_r + 0.3420\,\hat{a}_\theta\right)$$

$$\vec{E} = 0.8430\left(1.8792\,\hat{a}_r + 0.342\,\hat{a}_\theta\right)$$

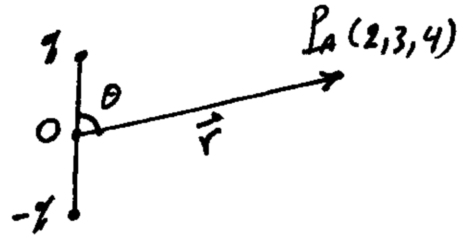
$$\vec{E} = 1.5841\,\hat{a}_r + 0.2883\,\hat{a}_\theta \ \ (V/m).$$

Ex. 2 : An electric dipole located at the origin in free space has a moment $\vec{p} = 3\hat{i} - 2\hat{j} + \hat{k}$ nC.m.

(a): Find V at $P_A(2,3,4)$ m.

(b): Find V at $r = 2.5$ m, $\theta = 30°$.

Sol.: (a): $\vec{r} = 2\hat{i} + 3\hat{j} + 4\hat{k}$



Using: $V = \dfrac{\vec{p} \cdot \vec{r}}{4\pi\epsilon_0 r^3}$

$\vec{p} \cdot \vec{r} = (3\hat{i} - 2\hat{j} + \hat{k}) \cdot (2\hat{i} + 3\hat{j} + 4\hat{k}) = 6 - 6 + 4$

$\vec{p} \cdot \vec{r} = 4 \text{ nC.m}^2$

$r = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{4 + 9 + 16} = \sqrt{29}$ m

$$V = \frac{4 \times 10^{-9}}{4 \times 3.14 \times 8.854 \times 10^{-12} \times 29\sqrt{29}} = \frac{4 \times 10^{3}}{17367.053}$$

$V = 0.2303$ V.

(b): Try to solve it yourself.

Ans.: $V \simeq 4.6621$ V.