1.4. Laplace Transform:

Several techniques used in solving engineering problems are based on the replacement of functions of real variable (usually time) by functions of a complex variable dependent up on frequency. The Laplace transform is very important transformation technique for linear control system analysis. Applications of this mathematical transformation are to solving linear constant – coefficient differential equation.

The Laplace transform is defined in the following manner:

$$F(s) = L\,[f\,(t)] = \int_0^{\infty} e^{-st}\ [f\,(t)]\,dt \ \ldots\ldots\ldots\ldots\ldots$$

Where:

f (t) is a function of time t such that f (t) =0 for t < 0 ,

f (s) is Laplace transform of f (t) ,

and the variable s is referred as the Laplace operator which is a complex variable , that is s = σ + jw ,where σ is the real component and w is the imaginary component .

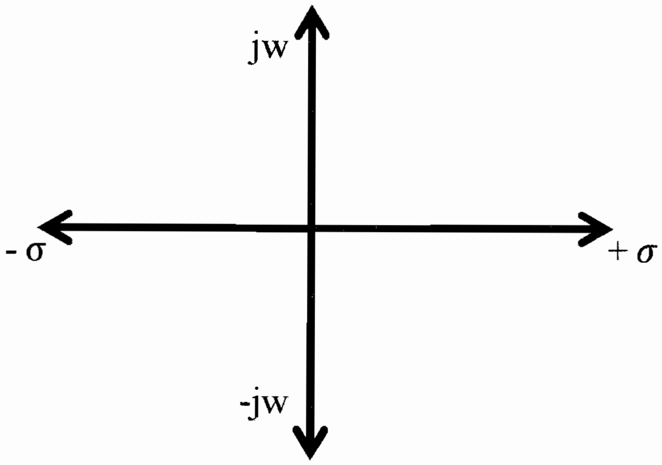

Fig (1-5) s-plane

Common functions

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The unit impulse function

$$\delta\ (t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

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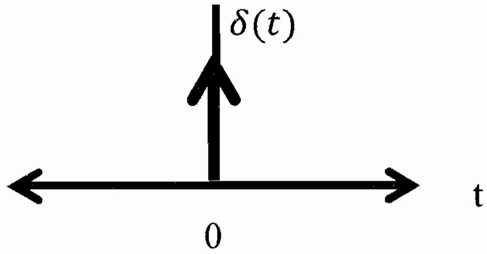

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The unit step function

$$u(t) = 0 \quad for\ t < 0$$
$$u(t) = 1 \quad for\ t > 0$$

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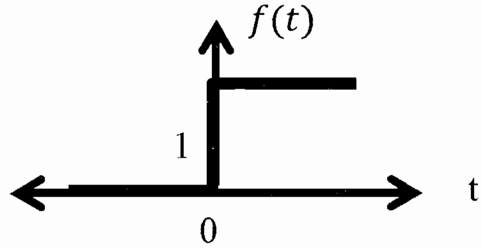

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The unit ramp function

$$f(t) = 0 \quad for\ t < 0$$
$$f(t) = 0 \quad for\ t \geq 0$$

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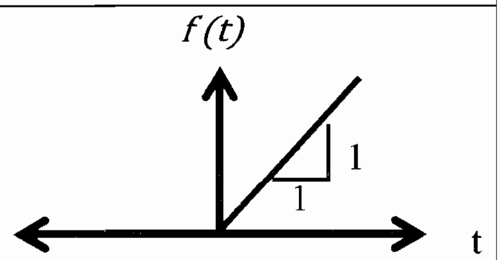

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The exponential function

F (t) =0 for t < 0
F (t) =e^{-at} for t > 0

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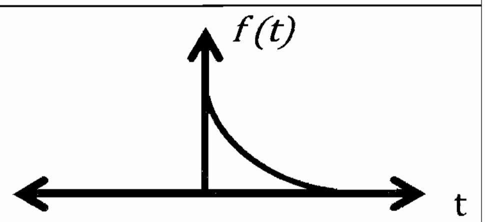

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The sinusoidal function

F (t) = 0 for t <0
F (t) = sin (wt) for t ≥ 0

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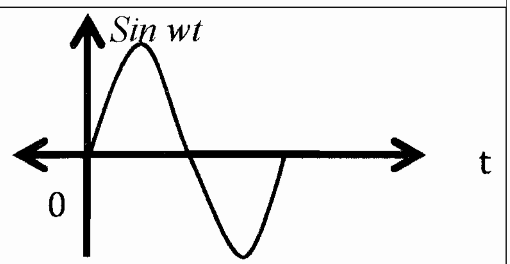

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Example

Find the Laplace transform f (s) of the unit step function defined by

$$f(t) = 0 \qquad \text{for } t < 0$$

$$f(t) = 1 \qquad \text{for } t > 0$$

Solution:

$$f(s) = l\,[f(t)] = \int_0^{\infty} 1.\, e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_0^{\infty} = -\frac{1}{s}[0 - 1] = \frac{1}{s}$$

Example

Find the Laplace transform f (s) of the exponential function defined by

$$f(t) = 0 \qquad \text{for } t < 0$$

$$f(t) = e^{-at} \qquad \text{for } t > 0$$

Solution:

$$f(s) = l\,[f(t)] = \int_0^{\infty} e^{-at} e^{-st} dt = \int_0^{\infty} e^{-(s+a)t}\, dt$$

$$= -\frac{1}{s+a} e^{-(s+a)t} \Big]_0^{\infty} = -\frac{1}{s+a}[0 - 1] = \frac{1}{s+a}$$

Table 1.1 Laplace transform pairs

Time function		Laplace transform
Unit impulse	δ(t)	1
Unit step	u(t)	$\frac{1}{s}$
Unit ramp	t	$\frac{1}{s^2}$
Polynomial	t^n	$\frac{n!}{s^n + 1}$
Exponential	e^{-at}	$\frac{1}{s + a}$
Sine wave	sin wt	$\frac{w}{s^2 + w^2}$
Cosine wave	cos wt	$\frac{s}{s^2 + w^2}$

1.5 Properties of Laplace transform

The Laplace transform and its inverse have several important properties which can be used advantageously in the solution of linear constant coefficient differential equations .they are:

1	$\mathcal{L}$ [A f(t)] = A f(s)
2	$\mathcal{L}\,[f_1(t) \mp f_2(t)] = F_1(s) \mp F_2(s)$
3	$\mathcal{L}\mp[\frac{d}{dt} f(t)] = s F(s) - \dot{f}(0\mp)$
4	$\mathcal{L}\mp[\frac{d^2}{dt^2} f(t) = s^2 F(s) - sf(0\pm) - \dot{f}(0\mp)$
5	$\mathcal{L}\mp[\int f(t)dt] = \frac{F(s)}{s} + \frac{1}{s}[\int f(t)dt]_{t=0\pm}$
6	$\mathcal{L}\mp[\int \dots \int f(t)\,(dt)^n] = \frac{F(s)}{s^n} + \sum_{k=1}^{n} \frac{1}{s^{n-k+1}} \frac{1}{s} [\int \dots \int f(t)(dt)^k]_{t=0\pm}$
7	$\mathcal{L}\,[\int_0^\infty f(t)\,dt] = \frac{F(s)}{s}$
8	$\int_0^\infty f(t)dt = \lim_{s\to 0} F(s)$ if $\int_0^\infty f(t)dt$ *exists*
9	$\mathcal{L}\,[e^{-at} f(t)] = F(s + a)$
10	$\mathcal{L}\,[f(t-\propto)\, 1(t-\propto)] = e^{-as} F(s)$ $\propto \geq 0$
11	$\mathcal{L}\,[t\, f(t)] = -\frac{d}{ds} F(s)$
12	$\mathcal{L}\,[t^2 f(t)] = \frac{d^2}{ds^2} F(s)$
13	$\mathcal{L}\,[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s)$ n = 1,2,3,…….
14	$\mathcal{L}\,[\frac{1}{t} f(t)] = \int_s^\infty F(s)\, ds$ if $\lim_{t\to 0} \frac{1}{t}$ f (t) exist
15	$\mathcal{L}\,[f(\frac{t}{a})] = a\, F(as)$
16	$\mathcal{L}\,[\int_0^t f_1(t-\tau)\, f_2(\tau)\, d\tau] = F_1(s)\, F_2(s)$
17	$\mathcal{L}\,[f(t)g(t)] = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F_1(p)\, G(s-p)\, dp$

18. the initial value f(0+) of the function f(t) whose laplace transform is F(s) is

$f(0+) = \lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s) \qquad t>0$

this relation is called the initial value theorem.

19. the final value f(∞) of the function f(t) whose laplace transform is F(s) is

$f(\infty) = \lim_{t\to\infty} f(t) = \lim_{s\to 0} s\,F(s)$

this relation is called the final value theorem.

1.6 Inverse Laplace Transform

Inverse laplace transform can be expressed as follows

$\mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j}\int F(s)\, e^{st}\, ds, \qquad \text{for } t > 0$

This expression is complicated for finding inverse of laplace transform thus there are two simpler methods as follows :

1. Use the table of laplace transform
2. Use partial fraction expression method

Partial fraction method

This technique uses partial fraction method to split up acomplicated fraction into forms that are in laplace transform table.

* Distinct Real Roots

Example

Find the inverse Laplace transform of

$$F(s) = \frac{s+1}{s(s+2)}$$

Solution

$$F(s) = \frac{s+1}{s(s+2)} = \frac{A_1}{s} + \frac{A_2}{s+2}$$

we can find the two unknown coefficient using the "cover - up" method

$$A_1 = \left.\frac{(s+1)\cancel{(s)}}{\cancel{(s)}(s+2)}\right|_{s=0} = \frac{1}{2}$$

$$A_2 = \frac{(s+1)\cancel{(s+2)}}{s\cancel{(s+2)}} = \frac{-1}{-2} = \frac{1}{2}$$

So,

$$F(s) = \frac{1}{2}\frac{1}{s} + \frac{1}{2}\frac{1}{s+2}$$

and

$$f(t) = \frac{1}{2} + \frac{1}{2}e^{-2t}, \quad t > 0$$

* Order of numerator Polynomial equals or greater order of denominator

Example

Find the inverse of Laplace transform of

$$F(s) = \frac{3s^2 + 2s + 3}{s^2 + 3s + 2}$$

Solution :

For the fraction shown above the order of the numerator POlynomial is not less than that of the denominator Polynomial.
There fore we must Perform long division.

$$\begin{array}{r|l} & \quad\quad\quad 3 \\ \cline{2-2} s^2 + 3s + 2 & 3s^2 + 2s + 3 \\ & 3s^2 + 9s + 6 \\ \cline{2-2} & \quad -7s - 3 \end{array}$$

Now, we can express the fraction as

a constant Plus a ProPer ratio of Polynomials.

$$F(s) = 3 + \frac{-7s - 3}{s^2 + 3s + 2}$$

$$= 3 + \frac{-7s - 3}{(s+1)(s+2)}$$

$$= 3 + \frac{A_1}{s+1} + \frac{A_2}{s+2}$$

Using the "cover-up" method we get

$A_1 = 4$

$A_2 = -11$

$$\therefore \quad F(s) = 3 + \frac{4}{s+1} - \frac{11}{s+2}$$

So,

$$f(t) = 3\,\delta(t) + 4\,e^{-t} - 11\,e^{-2t}$$

* Repeated Real Roots

Example

Find the inverse Laplace transform of

$$F(s) = \frac{s^2+1}{s^2(s+2)}$$

Solution :

$$F(s) = \frac{s^2+1}{s^2(s+2)} = \frac{A_1}{s+2} + \frac{A_2}{s} + \frac{A_3}{s^2}$$

we can find two of the unknown coefficients using the "cover-up" method.

$$A_1 = \left. \frac{(s^2+1)\cancel{(s+2)}}{s^2\cancel{(s+2)}} \right|_{s=-2} = \frac{5}{4}$$

$$A_3 = \left. \frac{(s^2+1)\cancel{s^2}}{\cancel{s^2}(s+2)} \right|_{s=0} = \frac{1}{2}$$

We find the other term using cross-

Multiplication:-

$$s^2+1 = s^2(s+2)\left[\frac{A_1}{(s+2)} + \frac{A_2}{s} + \frac{A_3}{s^2}\right]$$

$$= s^2 A_1 + s(s+2)A_2 + (s+2)A_3$$

Equating like Powers of "s" gives as:

$$1 = A_1 + A_2$$

$$0 = 2A_2 + A_3$$

$$1 = 2A_3$$

$$F(s) = \frac{5}{4}\frac{1}{s+2} - \frac{1}{4}\frac{1}{s} + \frac{1}{2}\frac{1}{s^2}$$

and ;

$$f(t) = \frac{5}{4}e^{-2t} - \frac{1}{4} + \frac{1}{2}t \ , \ t > 0$$

* Complex Roots

Example

Find the inverse Laplace transform of

$$F(s) = \frac{2s+12}{s^2+2s+5}$$

Solution :

$$s^2+2s+5 = (s+1+j2)(s+1-j2)$$

$$F(s) = \frac{2s+12}{s^2+2s+5}$$

$$= \frac{10+2(s+1)}{(s+1)^2+2^2}$$

$$= 5\,\frac{2}{(s+1)^2+2^2} + 2\,\frac{s+1}{(s+1)^2+2^2}$$

1.7 Solving Linear Constant - coefficient differential equations :

Solving linear constant coefficient differential equations by Laplace transform as shown in the following two examples

Example

Find the solution of $X(t)$ of the differential equation

$$\ddot{X} + 3\dot{X} + 2X = 0 \quad , \quad X(0) = a \quad , \quad \dot{X}(0) = b$$

where a and b are constants

Solution :

$$\mathcal{L}[\dot{X}] = s\,X(s) - X(0)$$

$$\mathcal{L}[\ddot{X}] = s^2 X(s) - s\,x(0) - \dot{x}(0)$$

The differential equation become

$$[s^2 X(s) - as - b] + 3[s X(s) - a] + 2X(s) = 0$$

$$(s^2 + 3s + 2)\,X(s) = as + b + 3a$$

$$X(s) = \frac{as+b+3a}{s^2+3s+2} = \frac{as+b+3a}{(s+1)(s+2)}$$

$$= \frac{2a+b}{s+1} - \frac{a+b}{s+2}$$

$$x(t) = L^{-1}[X(s)]$$

$$= L^{-1}\left[\frac{2a+b}{s+1}\right] - L^{-1}\left[\frac{a+b}{s+2}\right]$$

$$= (2a+b)e^{-t} - (a+b)e^{-2t}, \text{ for } t \geq 0$$

Example

Find the solution of $x(t)$ of the differential equation

$$\ddot{x} + 2\dot{x} + 5x = 3, \quad x(0) = 0, \quad \dot{x}(0) = 0$$

Solution

$$s^2 X(s) + 2s X(s) + 5X(s) = \frac{3}{s}$$

$$X(s) = \frac{3}{s(s^2+2s+5)}$$

$$x(t) = L^{-1}[X(s)]$$