Transient - Response Analysis

* Test signals :- In analysing and designing control systems we must have abasis of comparision of performance of various control systems. This basis may be set up by specifying particular test input signals and by coparing the response of various systems to these input signal.

The commonly used test input signals are those of step functions, ramp functions acceleration functions, impulse functions and sinusoidal functions.

if the input to control system are gradually changing functions of time, then a ramp function of time may be agood test signal.

if asystem is subjected to sudden disturbances a step function of time may be good test Signal.

if asystem is subjucted to shock inputs, an **impulse** function may be best.

The control system which is designed on the basis of test signals, the Performance of the system in response to actual inputs is generally satisfactory. The use of such test signals enables one to compare the Performance of all Systems on the same basis.

system time response :- it consists of two Part :

1) transient response :- response goes from initial state to final state.
2) steady state responce :- the manner in which the system output behaves as t approaches infinity.

* a control system is in equilibrium if, in the absence of any disturbance or input, the output stay in the same state.

* A linear time-invariant control system is stable if the output eventually comes back to its equilibrium state when the system is subjected to an initial condition

* A linear time-invariant control system is critically stable if oscillations of the output continue forever.

* obsolute stability, whether the system is stable or unstable.

* unstable system if the output diverges without bound from its equilibrium state when the system is subjected to an initial condition.

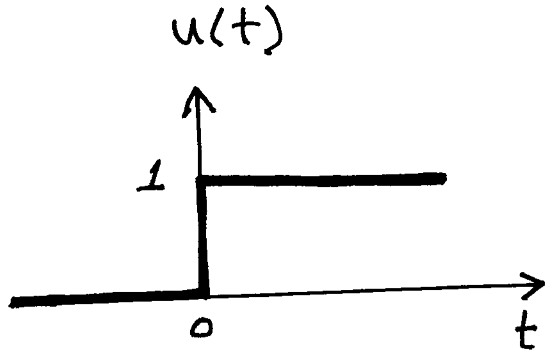
First order system

In this system the highest order of s in the denominator is 1

1- unit step response

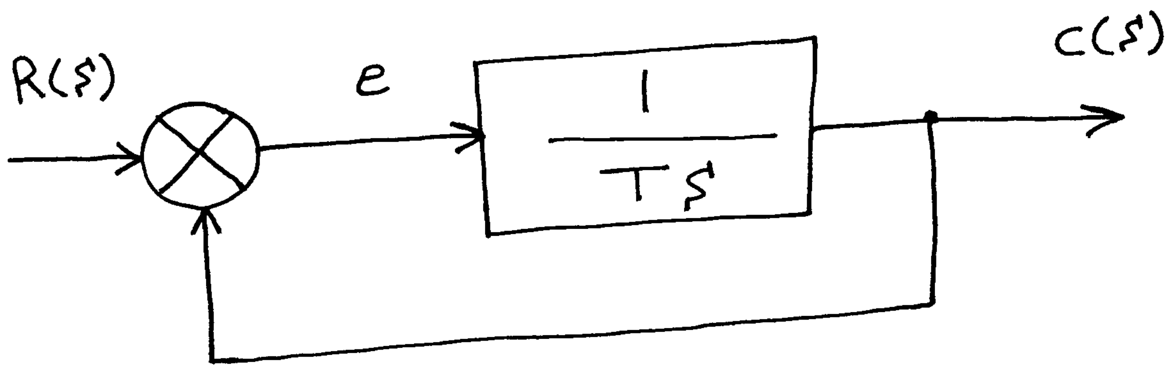
$u(t) = 1 \quad t \geq 0$

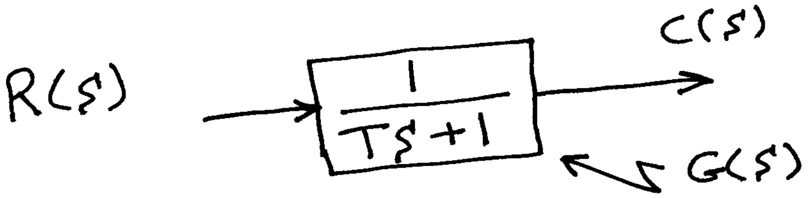
$u(t) = 0 \quad t < 0$



The laplace transform $\mathcal{L}\, u(t) = \frac{1}{s} = R(s)$

consider the first order system shown below





$$c(s) = G(s)\,R(s)$$

$$c(s) = \frac{1}{Ts+1} \cdot \frac{1}{s}$$

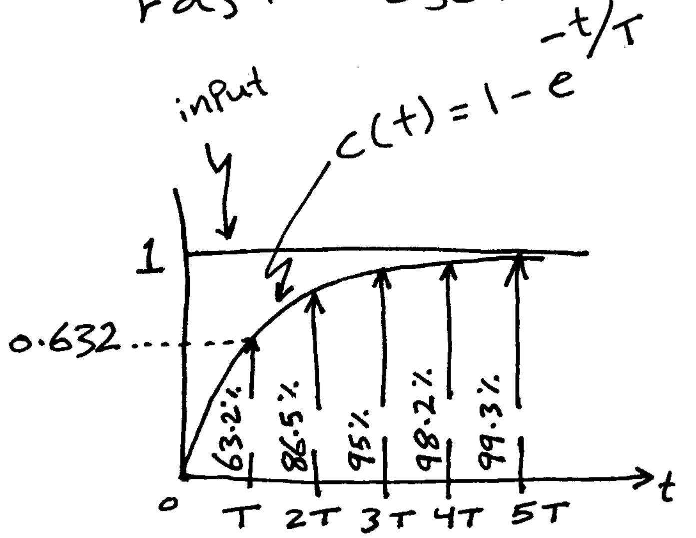
using Partial fraction expansion

$$c(s) = \frac{1}{s} - \frac{T}{Ts+1}$$

Taking the inverse Laplace transform

$$c(t) = 1 - e^{-t/T} \qquad t \geq 0$$

where T is the time constant of the system. The smaller time constant, the faster system response



So after 4T "four time constant" system will reach 98% from the final value.

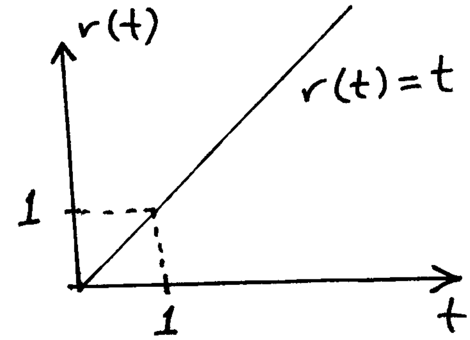
* The error is given by $e(t) = r(t) - c(t)$
$$= 1 - (1 - e^{-t/T})$$

* The steady state error $e_{ss} = \lim_{t \to \infty} e(t)$
$$= 1 - (1 - 0) = 0$$

2- Unit ramp response

$$r(t) = t \qquad t \geqslant 0$$

$$\mathcal{L}(r(t)) = R(s) = \frac{1}{s^2}$$

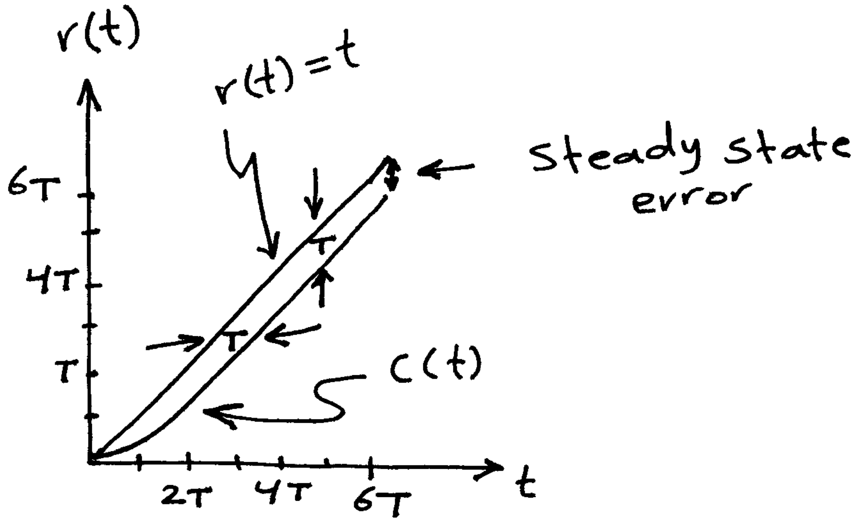


$$\therefore \; c(s) = \frac{1}{Ts+1} \cdot \frac{1}{s^2}$$

Using Partial fraction analysis

$$c(s) = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{Ts+1}$$

$$c(t) = \mathcal{L}^{-1} c(s) = t - T + T e^{-t/T}$$



The error signal $e(t) = t - (t - T + Te^{-t/T})$

$$= T(1 - e^{-t/T})$$

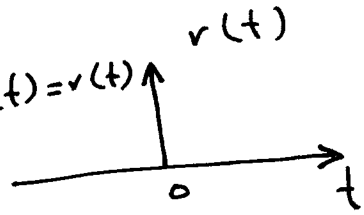
as t approach infinity $e^{-t/T}$ approach zero

$\therefore\ e(\infty) = ess =$ steady state error $= T(1 - e^{-\infty})$

$$e(\infty) = T$$

So, smaller the time constant smaller the steady state error.

3- Unit impulse response

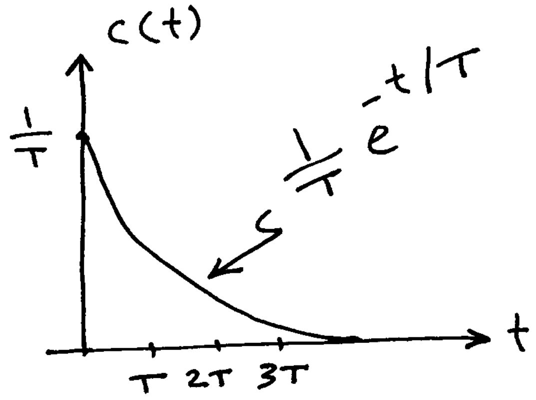


$$r(t) = \delta(t)$$

$$R(s) = \mathcal{L}\,\delta(t) = 1$$

$$C(s) = \frac{1}{Ts+1} = \frac{\frac{1}{T}}{s + 1/T}$$

$$c(t) = \mathcal{L}\,C(s) = \frac{1}{T}e^{-t/T}$$



$$c(t) = t - T + Te^{-t/T} \qquad t \geq 0$$

by differentiation of c(t) for ramp, we can get the response to unit step

$$c(t) = 1 - e^{-t/T} \qquad t \geq 0$$

by differentiation of c(t) again, we can get the response to unit impulse

$$c(t) = \frac{1}{T} e^{-t/T}$$

So, the response to the derivative of an input signal can be obtained by differentating the response of the original signal.

higher order system

Consider the following

$$Y(s) = \text{out Put} = \frac{A(s)}{B(s)}$$

a) B(s) contain Distinct roots :-

when the denominator function B(s) contain distinct roots only. Then it can be factored in the form

$$B(s) = (s - r_1)(s - r_2) \cdots (s - r)$$

using Partial fraction analysis, we get

$$Y(s) = \frac{K_1}{(s - r_1)} + \frac{K_2}{(s - r_2)} + \cdots + \frac{K_i}{(s - r_i)}$$

The constants K_i can be evaluated by

$$K_i = \lim_{s \to r_i} [(s - r_i)\, Y(s)]$$

The inverse transformation of $Y(S)$ gives

$$y(t) = K_1 e^{r_1 t} + K_2 e^{r_2 t} + \dots + K_i e^{r_i t}$$

* it is clear from above equ. that each root $r_1, r_2, \dots, r_i$ must be negative in order that each term $K_i e^{r_i t}$ in $y(t)$ be decaying function "stable system"

* if any root is +ve then $y(t)$ will increase with out bounds as t increase to infinity

* if $r_i = 0$, a constant term results because

$$K_i e^{0t} = K_i$$

b) $B(s)$ contain Repeated roots

$B(s)$ has a multiple or repeated roots which occurs q times. $B(s)$ may be factored in the form

$$B(s) = (s-r)^{q}(s-r_1)(s-r_2)\cdots(s-r_n)$$

The corresponding partial fraction expansion of $Y(s)$ is

$$Y(s) = \frac{C_q}{(s-r)^q} + \frac{C_{q-1}}{(s-r)^{q-1}} + \cdots + \frac{C_1}{s-r} + \frac{K_2}{s-r_2} + \cdots + \frac{K_n}{s-r_n}$$

The constant cofficients for the multiple terms are evaluated as follows

$$C_{q-k} = \lim_{s\to r}\left[\frac{1}{k!}\frac{d^k}{ds^k}(s-r)^{q}\,Y(s)\right]$$

Then from the transform table, the inverse transform is found to be

$$y(t) = \left[\frac{C_q t^{q-1}}{(q-1)!} + \frac{C_{q-1} t^{q-2}}{(q-2)!} + \dots + \frac{C_2 t}{1!} + C_1\right] e^{rt} + K_1 e^{r_1 t} + \dots$$

* if the value of r is Positive, $y(t)$ will become infinite as time increases.

* For $-ve$ r, a decreasing exponential results and thus the response term due to the repeated roots eventually vanishes.

Complex Conjagate roots

Complex roots, of $B(s)$ always occurs in Pairs, and furthermore these roots are always conjuagates of one another.

$B(s)$ has a complex root $a+jb$, then $a-jb$ will also be a root of $B(s)$. General form

$$s^2 - 2as + (a^2 + b^2) = [cs - (a+jb)][s - (a-jb)]$$

EX:- $s^2 + 4s + 9 \Rightarrow -2a = 4 \quad \therefore a = -2$

$$\Rightarrow a^2 + b^2 = 9$$

$$(-2)^2 + b^2 = 9$$

$$b^2 = 5 \Rightarrow b = \sqrt{5}$$

Thus the complex conjugate roots are

$$a \pm jb = -2 \pm j\sqrt{5}$$

if b is found to be an imaginary number, then the two roots are real and unequal.

EX:- $s^2+8s+12 \Rightarrow -2a=8 \Rightarrow a=-4$

$$a^2+b^2=12 \Rightarrow (-4)^2+b^2=12$$

$$b^2=-4 \Rightarrow b=J2$$

$$\therefore\ a \pm jb = -4 \pm j(j2) = -4 \mp 2 = -6, -2$$

are real & unequal

For complex conjugate roots $B(s)$ may be factored as

$$B(s) = (s-a-jb)(s-a+jb)(s-r_1)\cdots(s-r_{n-2})$$

The solution for $y(t)$ after using partial fraction and inverse transformation is

$$y(t) = \frac{1}{b}\left|K(a+jb)\right| e^{at}\sin(bt+\alpha) + K_1 e^{r_1 t} + \cdots K_{n-2} e^{r_{n-2}t}$$

where

$$K(a+jb) = \left[(s^2-2as+a^2+b^2)\frac{A(s)}{B(s)}\right]_{s=a+jb}$$

α = angle of $K(a+jb)$

Example:- Determine the inverse transformation of the following transformed equation.

$$Y(s) = \frac{75}{(s^2+4s+13)(s+6)}$$

Solution:-

for the complex conjugate roots

$$s^2+4s+13=0$$

$$-2a=4 \Rightarrow a=-2$$

$$a^2+b^2=13 \Rightarrow b^2=9 \Rightarrow b=3$$

so, $a \pm jb = -2 \pm j3$

$$K(a+jb) = \left[(s^2+4s+13)\frac{75}{(s^2+4s+13)(s+6)}\right]_{s=-2+j3}$$

$$K(a+jb) = \frac{75}{4+j3} = 15\angle -36.9^\circ$$

Thus $|K(a+jb)| = 15$, $\alpha = \angle K(a+jb) = -36.9$

The general form of inverse transformation is

$$y(t) = \frac{1}{b}\left|K(a+jb)\right| e^{at} \sin(bt+\alpha) + K_1 e^{r_1 t}$$

$$K_1 = \lim_{s \to -6} \frac{75}{s^2+4s+13} = \frac{75}{25} = 3$$

Thus, the desired result

$$y(t) = 5e^{-2t} \sin(3t - 36.9) + 3e^{-6t}$$

$$\cancel{y(t) = 5e^{-2t} \sin(3t - 36.9) + 3e}$$

Apply $\sin(\alpha+\beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$

$$\alpha = 3t \quad , \quad \beta = -36.9$$

yields the alternate form

$$y(t) = e^{-2t}(4\sin 3t - 3\cos 3t) + 3e^{-6t}$$

This result may be obtained as follows

$$Y(s) = \frac{75}{(s^2+4s+13)(s+6)} = \frac{As+B}{(s+2)^2+3^2} + \frac{K_1}{s+6}$$

Evaluation of the constants yield

$A = -3$, $B = 6$ & $K_1 = 3$

$$\therefore Y(s) = 4 \frac{3}{(s+2)^2+3^2} - 3 \frac{s+2}{(s+2)^2+3^2} + \frac{3}{s+6}$$

Inverting yields the Preceding form of $y(t)$

Now: consider again the following equation

$$y(t) = \frac{1}{b} \left| K(a+jb) \right| e^{at} \sin(bt+\alpha) + K_1 e^{r_1 t} + \dots + K_{n-2} e^{r_{n-2} t}$$

* The first term represents the exponentially damped sinusoidal results from complex conjugate roots of $B(s)$

* exponential factor a is the real of the complex conjugate roots

* Imajinary Part b is the frequency of oscilation of the exponentially damped Sinusoidal.

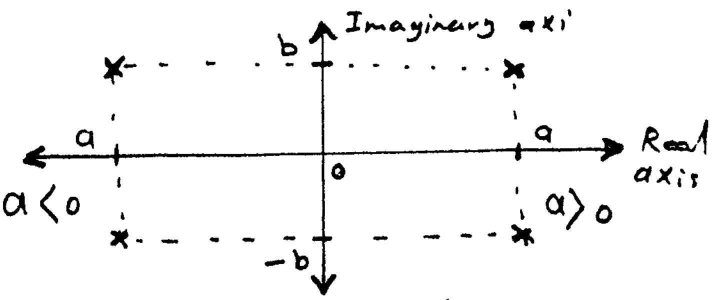
b = damped natural frequency. with $\frac{2\pi}{b}$ Period of each oscillation.

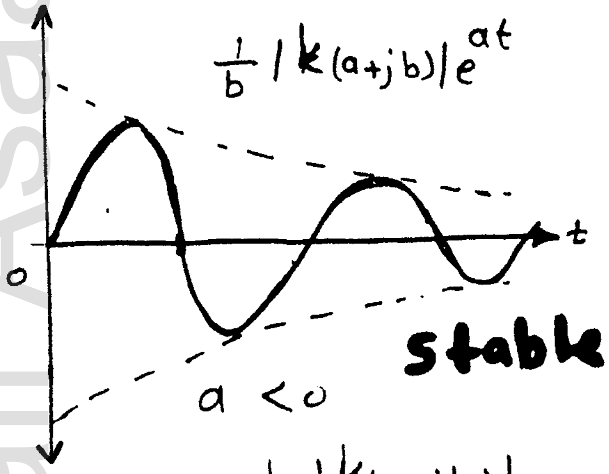
* $\frac{1}{b}|K(a+jb)|e^{at}$ = envelope of this Sinusoidal

if α is negative, the exponetial decreasing with time

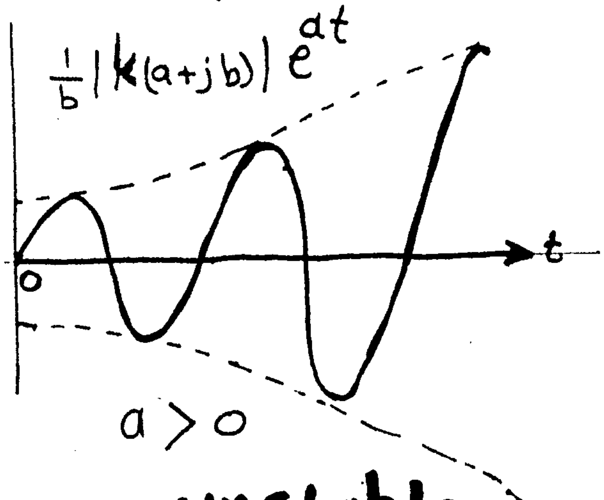
* if a is Positive the exponential increasing with time if $a=0$, Sinusoidal with constant amplitude results as $\frac{1}{b}|K(a+jb)|$

see figures below

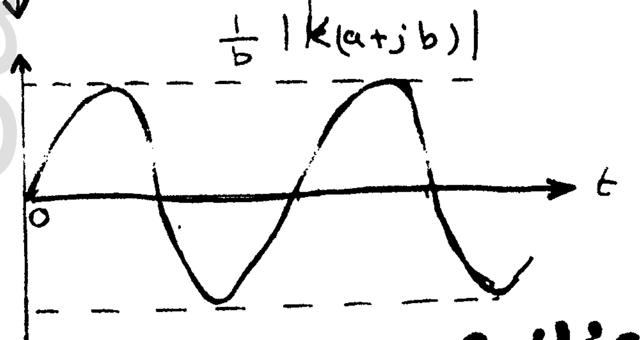




Stable

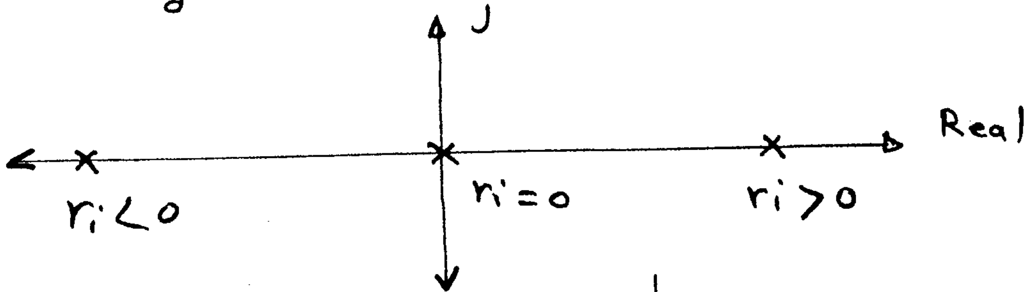


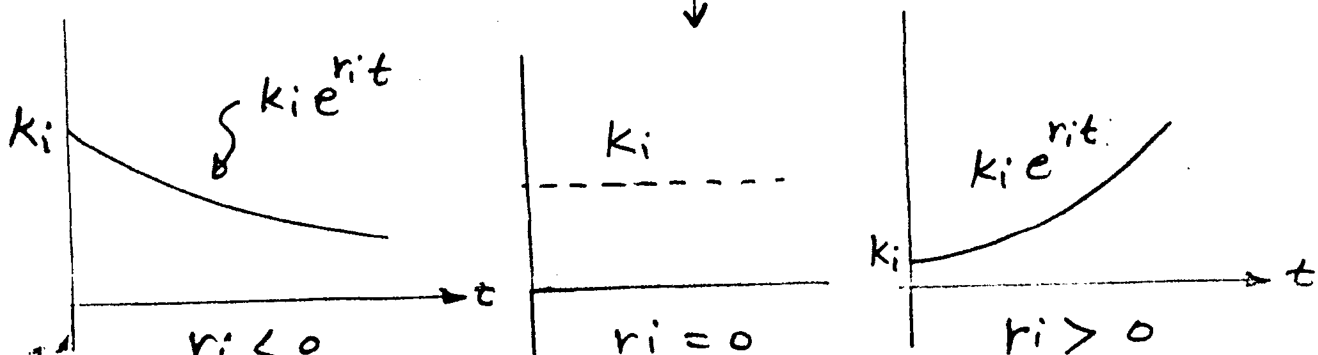
unstable



$a = 0$ **Critical stable**

The response terms types that result from real are shown in fig below





$r_i < 0$ stable

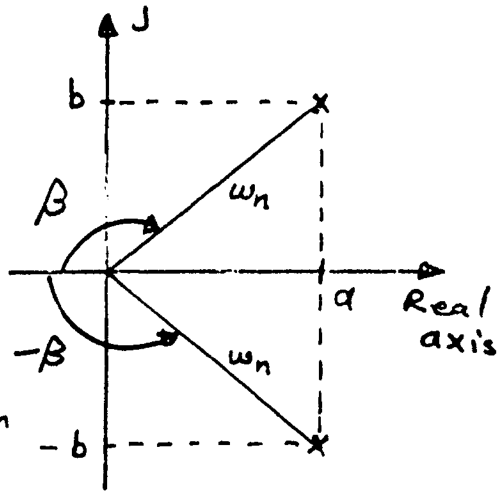
$r_i = 0$ critical stable

$r_i > 0$ unstable

Damping ratio and Natural frequency

In figure below, a complex conjugate roots is shown

The distance from each root to the origin $= \omega_n = \sqrt{a^2+b^2}$



The angle β is such that

$$\cos(\pi-\beta) = -\cos\beta = \frac{a}{\omega_n}$$

In term of polar coordinate β & ω_n

$$s^2 - 2as + a^2+b^2 = s^2 + (2\omega_n\cos\beta)s + \omega_n^2$$

$$a = -\omega_n\cos\beta \quad ; \quad \omega_n = \sqrt{a^2+b^2}$$

$2\omega_n\cos\beta$ is a measure of amount of damping in the system.

The response ceases to be sinusoidal when $\beta = 0$. For this critically damped case, the quadratic form is

$$s^2 + 2\omega_n s + \omega_n^2 = (s+\omega_n)^2$$

Thus the roots are repeated.

Damping ratio :- The ratio of the actual amount of damping $(2\omega_n\cos\beta)$ to the critical amount of damping $(2\omega_n)$

$$\text{Damping ratio} \quad \zeta = \frac{2\omega_n\cos\beta}{2\omega_n} = \cos\beta$$

Then as β goes from 0 to π ; ζ goes from 1 to -1

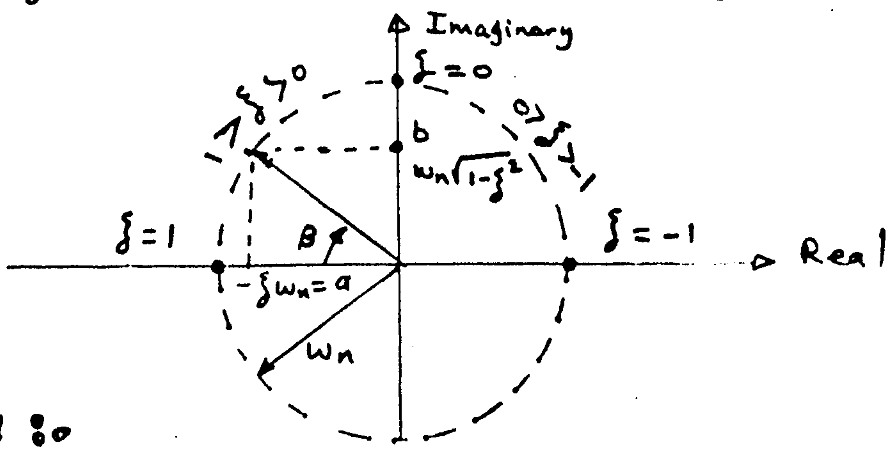
note: $0 < \beta < \frac{\pi}{2} \Rightarrow$ roots in the left hand side of s-plane.
$0 < \zeta < 1 \Rightarrow$ yield sinusoidal decreasing with time.

note: $\frac{\pi}{2} < \beta < \pi \Rightarrow$ roots in the right hand side of s-plane

$-1 < \zeta < 0 \Rightarrow$ yield sinusoidal increasing with time

In term of ζ, the quadratic form becomes

$$s^2 + 2\zeta\omega_n s + \omega_n^2 \quad , \quad s_1, s_2 = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$



Example:

$$s^2 + 4s + 9$$

$$\omega_n^2 = 9 \Rightarrow \omega_n = \pm 3 \Rightarrow \omega_n = 3$$

ω_n must be positive because it represent the distance from the origin to the root.

$$2\zeta\omega_n = 4 \Rightarrow 2 * 3\zeta = 4 \Rightarrow \zeta = \frac{2}{3}$$

$$a = -\zeta\omega_n = -2 \quad , \quad b = \omega_n\sqrt{1-\zeta^2} = \sqrt{5}$$

The time response due to complex conjugate poles can be written in term of ζ & ω_n as

$$y(t) = \frac{1}{b}\left|K(a+jb)\right| e^{-\zeta\omega_n t} \sin\left(\omega_n\sqrt{1-\zeta^2}\, t + \alpha\right) + k_1 e^{r_1 t} + \cdots$$

if $\zeta = 0$, the response is

$$y(t) = \frac{1}{b}\left|K(a+jb)\right| \sin(\omega_n t + \alpha) + k_1 e^{r_1 t} + \cdots$$

which represent a sinusoidal with constant amplitude $\frac{1}{b}\left|K(a+jb)\right|$ and natural frequency of oscillation ω_n.

Finally, when the value of $\zeta > 1$, two real poles result rather than complex conjugate poles.

Example:- Find the general transient response of 2nd order system to unit step function as input.

$$Y(s) = \frac{w_n^2}{s^2 + 2\zeta w_n s + w_n^2} \; R(s) \leftarrow \frac{1}{s}$$

Solution:-

general transient response is of the form

$$y(t) = \frac{1}{b} |K(a+jb)| e^{at} \sin(bt+\alpha) + K_1 e^{r_1 t}$$

$r_1 = 0$

$$K(a+jb) = \left.\frac{W_n^2}{s}\right|_{s=a+jb} = \frac{W_n^2}{a+jb} = \frac{W_n^2}{\sqrt{a^2+b^2} \angle \tan^{-1}\frac{b}{a}}$$

$$\text{Thus } |K(a+jb)| = \frac{W_n^2}{\sqrt{a^2+b^2}} = \frac{W_n^2}{W_n} = W_n$$

$$\alpha = -\tan^{-1}\frac{b}{a} = \tan^{-1}\frac{-W_n\sqrt{1-\zeta^2}}{-\zeta W_n} = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

The constant K_1 is evaluated as follow

$$K_1 = \lim_{s \to 0} \frac{W_n^2}{s^2 + 2\zeta W_n s + W_n^2} = 1$$

Thus the desired transient response is

$$y(t) = \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta w_n t} \sin\left[(w_n\sqrt{1-\zeta^2})t - \alpha\right] + 1$$

For the case in which $\zeta = 1$, the quadratic term $s^2 + 2w_n s + w_n^2 = (s+w_n)^2$

Thus the Partial - fraction of $y(s)$ is

$$Y(s) = \frac{C_2}{(s+w_n)^2} + \frac{C_1}{s+w_n} + \frac{K_1}{s}$$

$$C_2 = \lim_{s \to w_n} \frac{w_n^2}{s} = -w_n$$

$$C_1 = \lim_{s \to -w_n} \frac{d}{ds} \frac{w_n^2}{s} = \left.\frac{-w_n^2}{s^2}\right|_{s=-w_n} = -1$$

$$K_1 = \lim_{s \to 0} \frac{w_n^2}{s+w_n^2} = 1$$

$$\therefore Y(s) = \frac{-w_n}{(s+w_n)^2} - \frac{1}{s+w_n} + \frac{1}{s}$$

The inverse transformation results

$$y(t) = -w_n t e^{-w_n t} - e^{-w_n t} + 1$$

$$y(t) = 1 - (w_n t + 1) e^{-w_n t}$$