Chapter Two

"Mathematical Models of systems"

2.1 Definitions

Linear time invariant systems (LTIS)

it is represented by differential equation whose cofficents are constants.

for example:-

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + X = 0$$

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + X = A \sin wt$$

Transfer function :- The transfer function of alinear, time-invariant differential equation system is defined as the ratio of Laplace transform of the out put (response function) to the Laplace transform of the input (driving function) under the assumption that all initial conditions are Zero.

Consider the linear time-invariant system

$$a_0 \overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1} \dot{y} + a_n y$$
$$= b_0 \overset{(m)}{x} + b_1 \overset{(m-1)}{x} + \cdots + b_{m-1} \dot{x} + b_m x$$

where :-

$n \geq m$

y : is the out Put of system

x : is the inPut of system.

$$\text{Transfer function} = G(s) = \left. \frac{\mathcal{L}[\text{output}]}{\mathcal{L}[\text{input}]} \right|_{\substack{\text{Zero} \\ \text{initial} \\ \text{condition}}}$$

$$= \frac{Y(s)}{X(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

2.2. Modeling of Electrical system :-

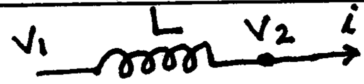
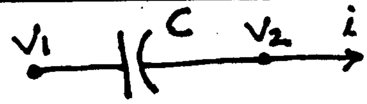
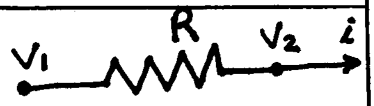
Basic laws governing electrical circuits are Kirchhoff's current law and Voltage law.

* Kirchhoff's current law (node law) states that the algebraic sum of all currents entering and Leaving anode is Zero
* Kirchhoff's Voltage law (Loop law) states that at any given instant the algebraic sum of Voltages arount any loop in an electrical circuit is Zero.

Amathematical model of an electrical circuit can be obtained by applying one or both of Kirchhoff's Law to it.

The electrical system elements and their symbols are listed in table(2.1)

Table (2.1) Electrical system Elements

Physical Element	Governing Equation	Symbol
Inductor	$v_{21} = L \frac{di}{dt}$	
Capacitor	$v_{21} = \frac{1}{C} \int i \, dt$	
Resistor	$v_{21} = R i$	

ملاحظة لأن المعادلات الرياضية التي تقود الدوائر الكهربائية هي معادلات تفاضلية فيجب أستخدام تحويل لابلاس لغرض التعامل مع هذه المعادلات .

لذلك يمكن كتابة علاقات الفولتية لكل من الملف المتسعة والمقاومة كما يلي :-

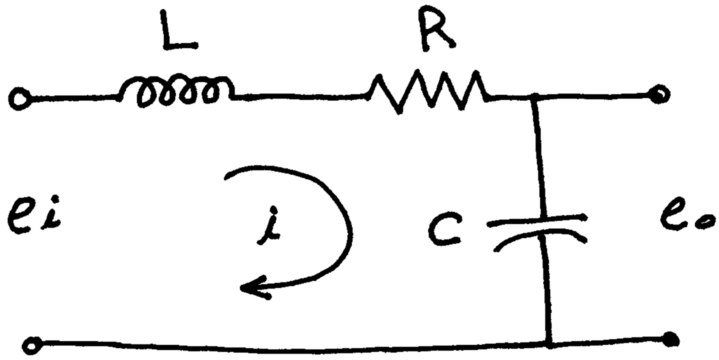
$$V_R = i R \longrightarrow V_R = I R$$

$$V_L = L \frac{di}{dt} \longrightarrow V_L = L S I$$

$$V_C = \frac{1}{C} \int i \, dt \longrightarrow V_C = \frac{1}{CS} I$$

Example :

obtain the transfer function $E_o(s)/E_i(s)$ for the electrical system shown in fig below.



Solution :

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = e_i$$

$$\frac{1}{C}\int i\,dt = e_o$$

Taking the Laplace transform, assuming zero initial conditions, we obtain:-

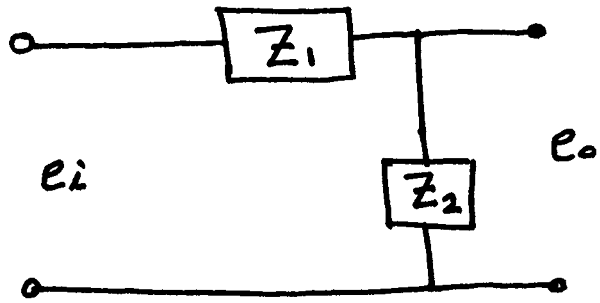
$$Ls\,I(s) + R\,I(s) + \frac{1}{Cs}\,I(s) = E_i(s)$$

$$\frac{1}{Cs}\,I(s) = E_o(s)$$

$$T.F = \frac{E_o(s)}{E_i(s)} = \frac{1}{LCs^2 + RCs + 1}$$

another Solution for Example

using complex impedance



$$Z_1 = Ls + R$$

$$Z_2 = \frac{1}{cs}$$

$$E_i = R\,I(s) + Ls\,I(s) + \frac{1}{cs}\,I(s)$$

$$E_o = \frac{1}{cs}\,I(s)$$

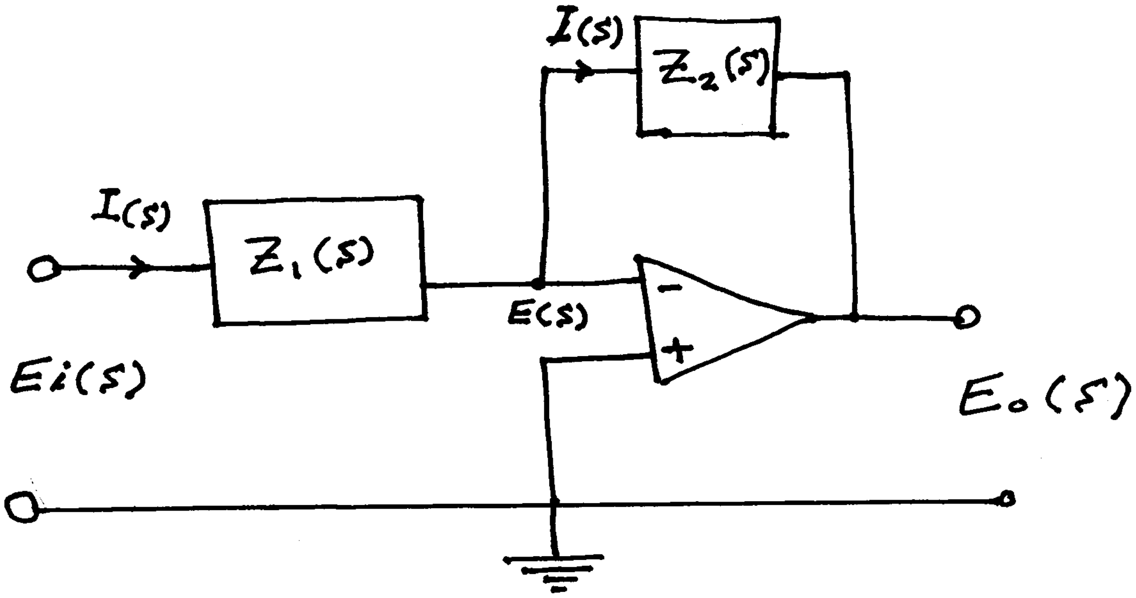
$$\therefore \quad \frac{E_o(s)}{E_i(s)} = \frac{\frac{1}{cs}}{Ls + R + \frac{1}{cs}} \times \frac{cs}{cs}$$

$$= \frac{1}{LCs^2 + RCs + 1}$$

2.3. Modeling of electronic system

operational amplifier :-

operational amplifier, often is called op-amps are frequently used to amplify signals.

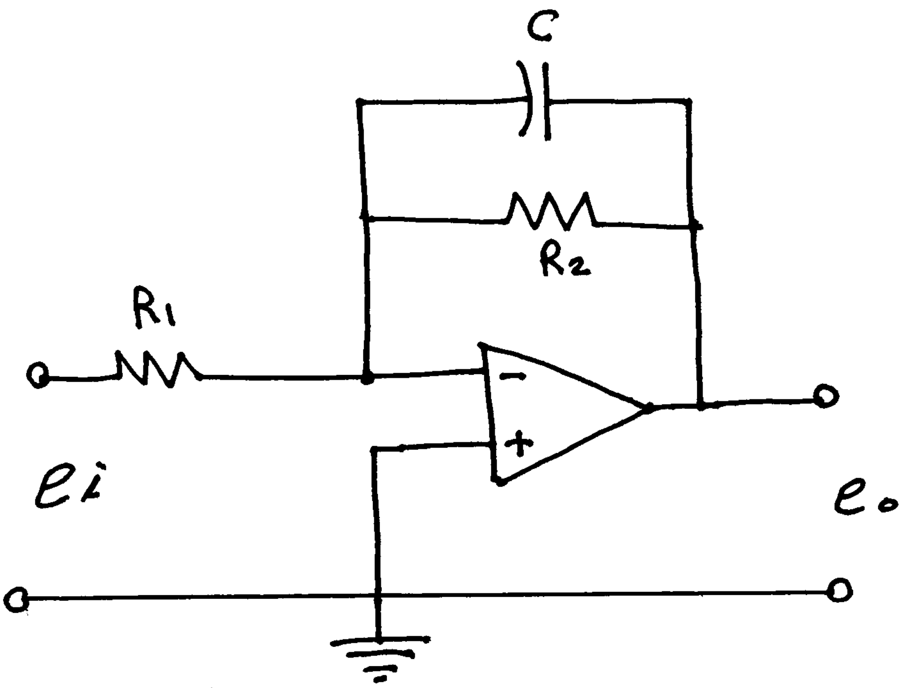


operational amplifier circuit

$$\frac{E_o(s)}{E_i(s)} = -\frac{Z_2(s)}{Z_1(s)}$$

Example 8:

obtain the transfer function $E_o(s)/E_i(s)$ of the electronic system shown in the figure below



Solution 8:

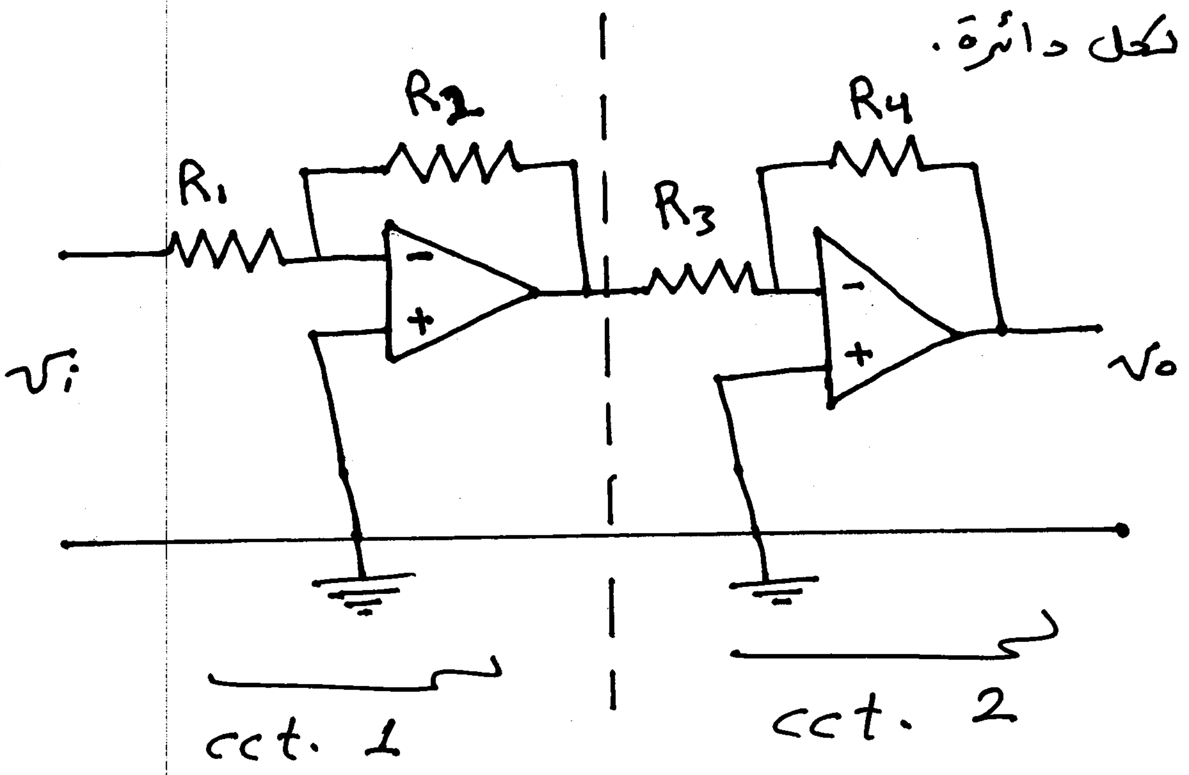
$$Z_1(s) = R_1 \quad \text{and} \quad Z_2(s) = \frac{1}{Cs + \frac{1}{R_2}}$$

$$= \frac{R_2}{R_2 C s + 1}$$

The transfer function $E_o(s)/E_i(s)$ is

$$T.f = \frac{E_o(s)}{E_i(s)} = \frac{-Z_2(s)}{Z_1(s)} = \frac{-R_2}{R_1} \frac{1}{R_2 C s + 1}$$

ملاحظة :- في حالة كون الدائرة الكهربائية تتكون من أكثر من دائرة فأن دالة الانتقال (Transfer function) ستكون حاصل ضرب دوال الانتقال (Transfer function) لكل دائرة .

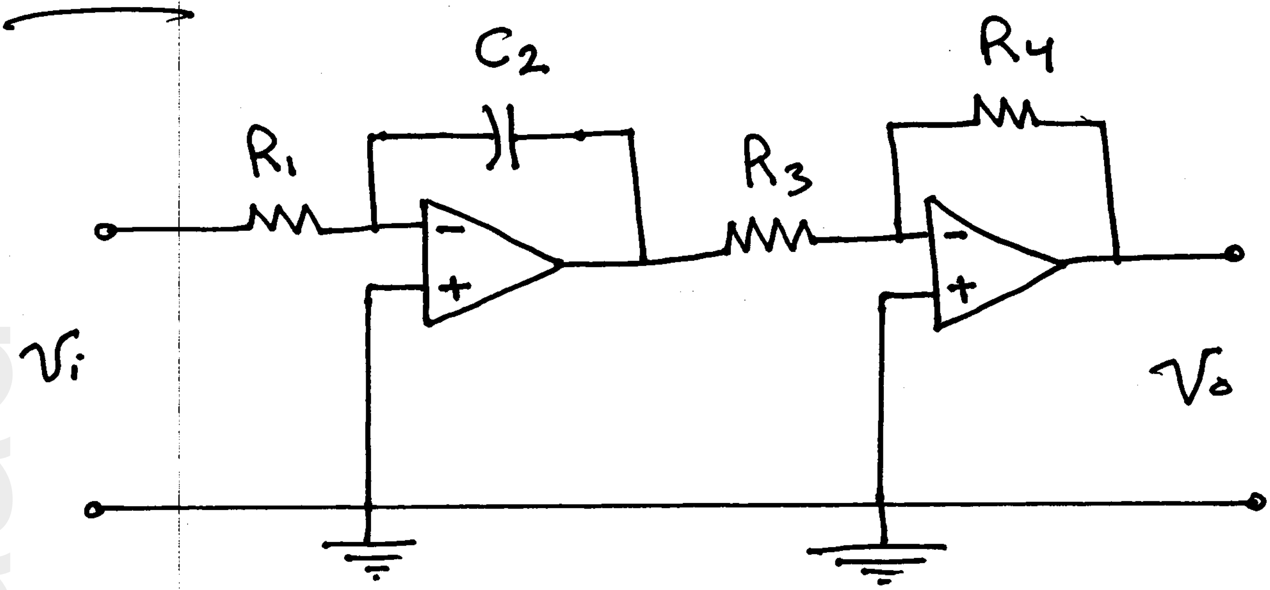


لايجاد $\frac{V_o}{V_i}$ نقوم بضرب دالة التحويل (Transfer function) للدائرة الأولى بدالة التحويل (Transfer function) للدائرة الثانية .

$$\frac{V_o}{V_i} = \left(-\frac{R_2}{R_1}\right) \times \left(\frac{-R_4}{R_3}\right)$$

$$= \frac{R_2 \, R_4}{R_1 \, R_3}$$

EXample :



Find $\frac{E_o(S)}{E_i(S)}$ Transfer function ?

Solution :

$Z_1 = R_1$ $\qquad$ $Z_3 = R_3$

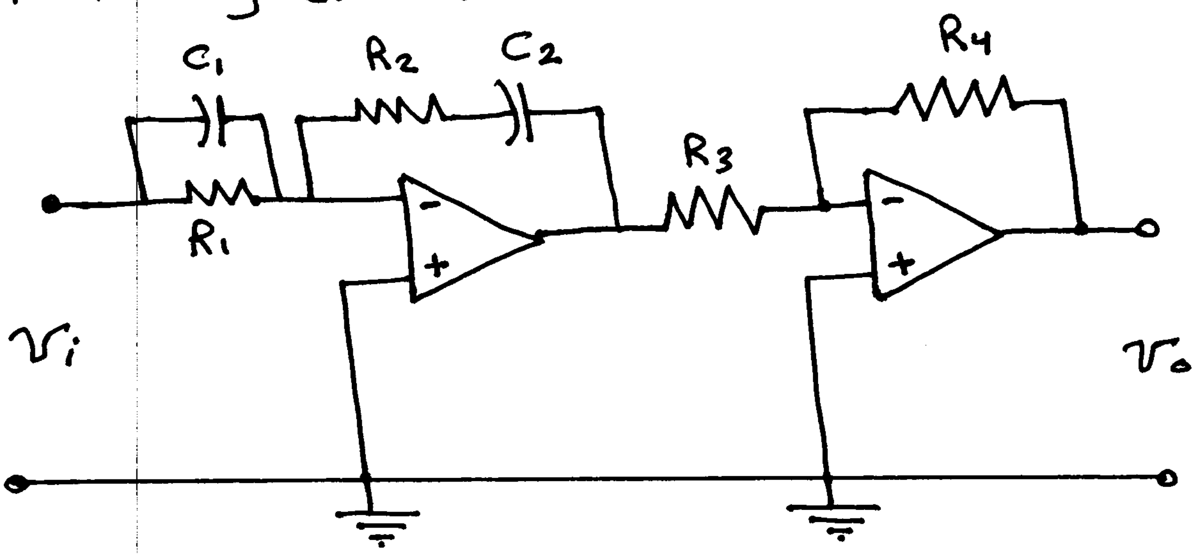
$Z_2 = \frac{1}{C_2 S}$ $\qquad$ $Z_4 = R_4$

$T.f_1 = \frac{-Z_2}{Z_1}$, $T.f_2 = \frac{-Z_4}{Z_3}$

$$T.f \text{ total} = \frac{Z_2 \; Z_4}{Z_1 \; Z_3} = \frac{R_4}{R_3} \; \frac{1}{R_1 C_2 S}$$

Example :

Find the transfer function for the following circuit



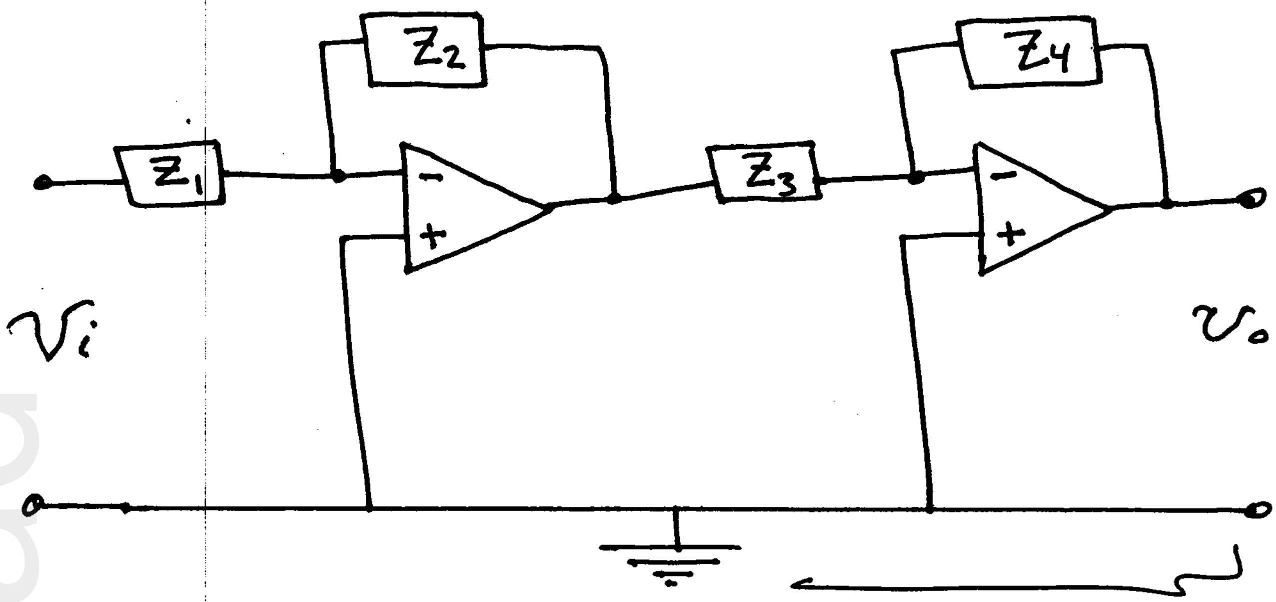
Solution :

$$Z_1 = \frac{1}{C_1 s} \parallel R_1 = \frac{R_1}{R_1 C_1 s + 1}$$

$$Z_2 = R_2 + \frac{1}{C_2 s} = \frac{R_2 C_2 s + 1}{C_2 s}$$

$$Z_3 = R_3$$

$$Z_4 = R_4$$



$$T.f_1 = -\frac{Z_2}{Z_1} \qquad T.f_2 = -\frac{Z_4}{Z_3}$$

$$\text{Total } T.f = \left(-\frac{Z_2}{Z_1}\right) \times \left(\frac{-Z_4}{Z_3}\right)$$

$$= \frac{Z_2 Z_4}{Z_1 Z_3}$$

$$= \frac{\frac{R_2 C_2 s + 1}{C_2 s} \times R_4}{\frac{R_1}{R_1 C_1 s + 1} \times R_3}$$

$$= \frac{R_2 R_4}{R_1 R_3} \frac{(R_1 C_1 s + 1)(R_2 C_2 s + 1)}{R_2 C_2 s}$$