2.8 Modeling in state space

State: The state of a dynamic system is the smallest set of variables (called state variable) such that the knowledge of these variables at $t = t_0$, together with the knowledge of input for $t \geq t_0$ completely determines the behavior of the system for any time $t \geq t_0$.

State vector: A state vector is a vector that determines uniquely the system state $X(t)$ for any time $t \geq 0$, once the state at $t = t_0$ is given and the input $U(t)$ for $t \geq t_0$ is specified.

State space : The n-dimensional space whose coordinate axis consist of the X_1 axis, X_2 axis, ---, X_n axis is called a state space

State space equations : In state-space analysis there are three types of variables that are involved in the modeling of dynamic system

1- Input variables
2- output variables
3- State variables.

* The number of state variables to completely define the dynamics of the system is equal to the number of integral involved in the system.

* The outputs of integrators serve as state variables

The equations of linearized system about the operating state are :-

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t)$$
$$y(t) = C(t)\,x(t) + D(t)\,u(t)$$

time-vary system

where :-

$A(t)$:- is the state matrix

$B(t)$:- is the input matrix

$C(t)$:- is the out put matrix

$D(t)$:- is the direct transmittance matrix

$x(t)$:- is the state vector

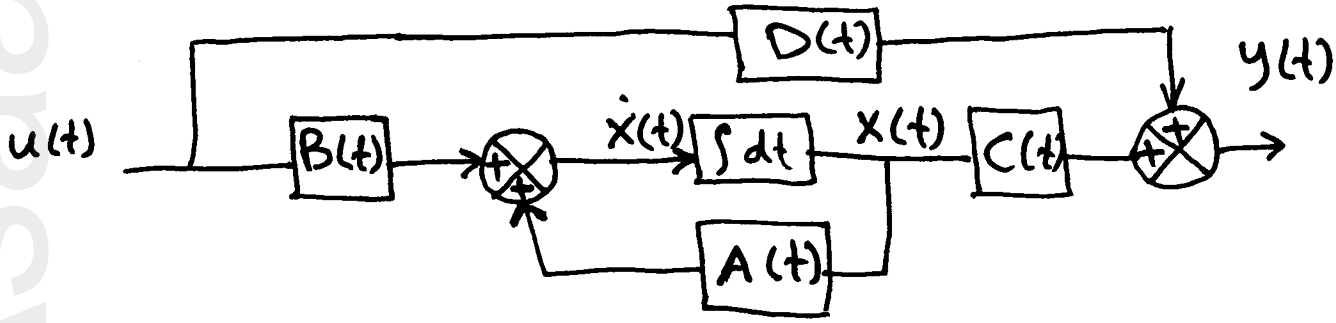
$u(t)$:- is the input vector

$y(t)$:- is the out put vector

* The state - space representation for time invariant system is shown below

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$

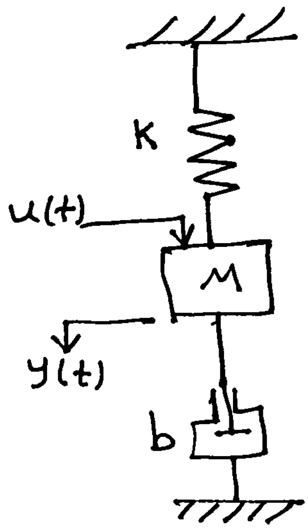
$$y(t) = C\,x(t) + D\,u(t)$$



Block diagram of the linear continuous time control system represented in state space.

Example : Consider the mechanical system shown below where $u(t)$ is the external force & $y(t)$ is the displacement caused by $u(t)$

Sol



the system equation is

$$m\ddot{y} + b\dot{y} + Ky = u$$

let us defined the state variable as :

$$x_1(t) = y(t)$$

$$x_2(t) = \dot{y}(t)$$

Then we obtain

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{1}{m}(-Ky - b\dot{y}) + \frac{1}{m}u$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{-K}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}u$$

The output equation is $\quad y = x_1$

In avector-Matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$$

The outputeq.
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

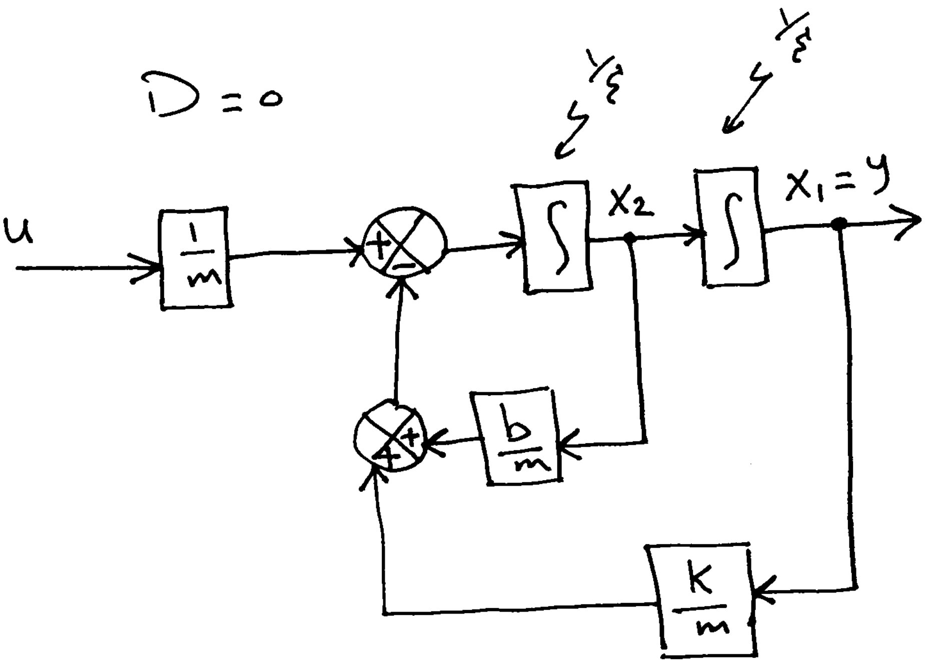
The standard formis

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

where $A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$

$D = 0$



How to drive the transfer function of single-input-single output system from the state-space equation

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Let us consider the system whose transfer function is given by

$$\frac{Y(s)}{U(s)} = G(s)$$

This system may be represented in state space by the following equations:-

$$\dot{x} = Ax + Bu \quad ---- (1)$$

$$y = Cx + Du \quad ---- (2)$$

The Laplace transform of state space equations are given by

$$s\,X(s) - X(0) = A\,X(s) + B\,U(s) \quad --- (3)$$

$$Y(s) = C\,X(s) + D\,U(s) \quad --- (4)$$

we assume $X(0)$ is Zero

$$s\,X(s) - A\,X(s) = B\,U(s)$$

or $$(sI - A)\,X(s) = B\,U(s)$$

or $$X(s) = (sI - A)^{-1}\,B\,U(s) \quad \text{---} (5)$$

by substituting eq. (5) into eq (4)

we get

$$Y(s) = \left[C\,[sI - A]^{-1}\,B + D\right]U(s)$$

$$\boxed{\therefore\ G(s) = C\,(sI - A)^{-1}B + D}$$

* in other words the eigen values of A are identical to the Poles of $G(s)$

Example: Obtain the transfer function for the system described by the state space equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

sol

zero

$$G(s) = C\left[sI - A\right]^{-1} B + D$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \left\{ \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \right\}^{-1} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ \frac{k}{m} & s+\frac{b}{m} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$

since

$$\begin{bmatrix} s & -1 \\ \frac{k}{m} & s+\frac{b}{m} \end{bmatrix} = \frac{1}{s^2 + \frac{b}{m}s + \frac{k}{m}} \begin{bmatrix} s+\frac{b}{m} & 1 \\ -\frac{k}{m} & s \end{bmatrix}$$

we have

$$G(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{s^2 + \frac{b}{m}s + \frac{K}{m}} \begin{bmatrix} s + \frac{b}{m} & 1 \\ -\frac{K}{m} & s \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$

$$G(s) = \frac{1}{ms^2 + bs + K}$$

Note

$$\begin{bmatrix} a & b \\ c & D \end{bmatrix}^{-1} = \frac{1}{\Delta} \begin{bmatrix} D & -b \\ -c & a \end{bmatrix}$$

Transfer Matrix :

Assume that there are n inputs $u_1, u_2, \ldots, u_n$ and m outputs $y_1, y_2, \ldots, y_m$

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \quad , \quad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

The transfer matrix $G(s)$ relates the output $Y(s)$ to the input $U(s)$ or

$$Y(s) = G(s)\, U(s)$$

State - Space representation of Dynamic system

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(1) State - Space representation of n-th order systems of Line differential equation in which the forcing function dose no involve derivative terms.

$$\overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \dots a_{n-1} \dot{y} + a_n y = u \quad \text{---} (1)$$

let us define

$$x_1 = y \quad , \quad x_2 = \dot{y} \quad , \quad \text{---} \; , \; x_n = \overset{(n-1)}{y}$$

Then equ. (1) can be written as

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\vdots$$
$$\dot{x}_{n-1} = x_n$$
$$\dot{x}_n = -a_n x_1 - \dots - a_1 x_n + u$$
$$\dot{x} = Ax + Bu$$

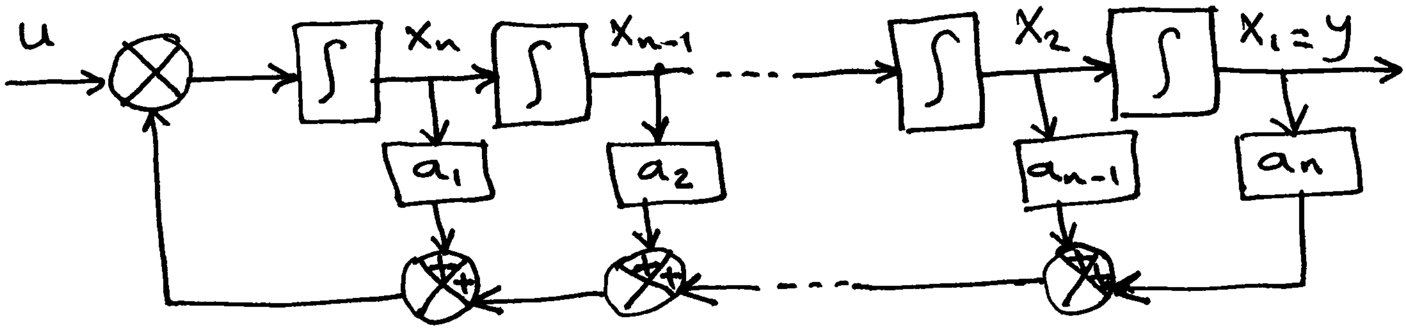
$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} , \quad A \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & & -a_1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

The output can be given by

$$y = [1 \quad 0 \quad \cdots \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} , \quad y = CX$$

$$C = [1 \quad 0 \quad \cdots \quad 0]$$



Example: consider the system defined by

$$\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = 6u$$

where y is the output and u is the input of the system obtain a state-space representation of the system.

Solution:

let us choose the state variable as

$$x_1 = y \quad , \quad x_2 = \dot{y} \quad , \quad x_3 = \ddot{y}$$

then we obtain

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -6x_1 - 11x_2 - 6x_3 + 6u$$

By use the vector notation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} [u]$$

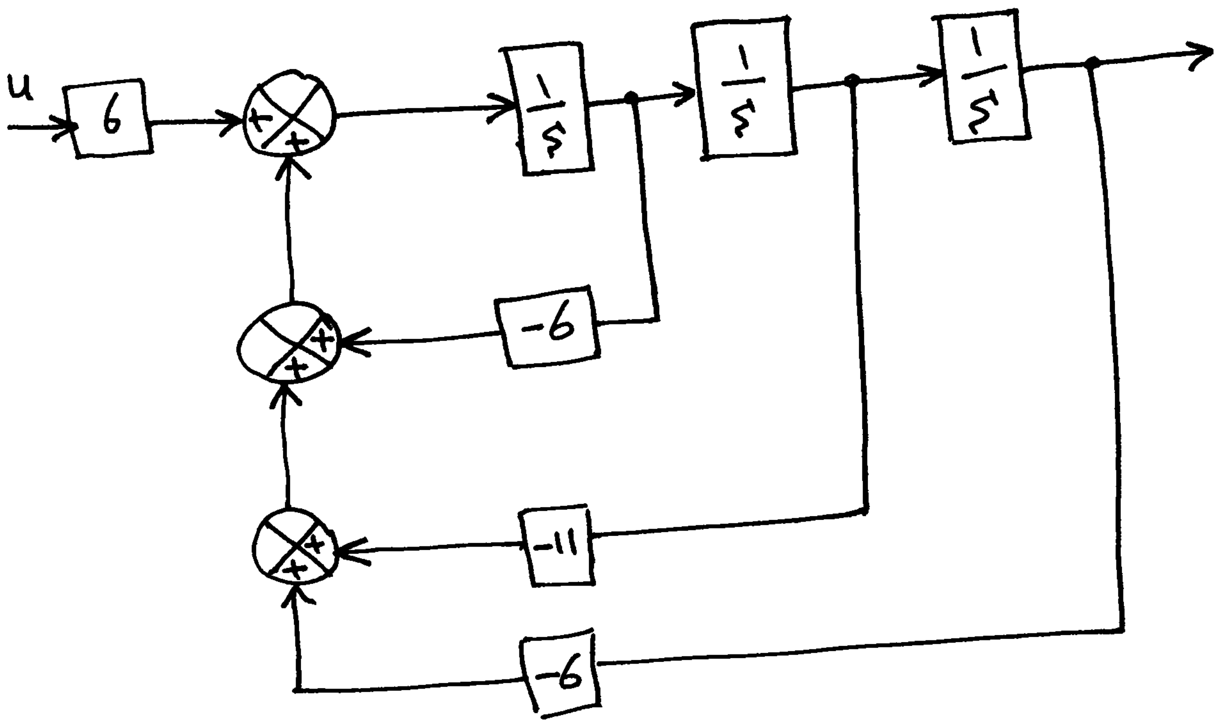
The out put equation is given by

$$y = [1 \; 0 \; 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The standard form is

$$\dot{x} = Ax + Bu$$

$$y = cx$$



State-space Representation of n-th order systems of Linear differential equations in which the forcing function involves derivative terms

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$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 u^{(n)} + b_1 u^{(n-1)} + \cdots + b_{n-1}\dot{u} + b_n u$$

let

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_n = -a_n x_1 - a_{n-1} x_2 - \cdots - a_1 x_n + b_0 u^{(n)} + b_1 u^{(n-1)} + \cdots + b_n u$$

$$x_1 = y - \beta_0 u$$

$$x_2 = \dot{y} - \beta_0 \dot{u} - \beta_1 u = \dot{x}_1 - \beta_1 u$$

$$x_3 = \ddot{y} - \beta_0 \ddot{u} - \beta_1 \dot{u} - \beta_2 u = \dot{x}_2 - \beta_2 u$$

$$\vdots$$

$$x_n = y^{(n-1)} - \beta_0 u^{(n-1)} - \beta_1 u^{(n-2)} - \cdots - \beta_{n-2}\dot{u} - \beta_{n-1} u$$

$$= \dot{x}_{n-1} - \beta_{n-1} u$$

where $\beta_0, \beta_1, \beta_2, \ldots, \beta_n$ are determined from

$$\beta_0 = b_0$$

$$\beta_1 = b_1 - a_1 \beta_0$$

$$\beta_2 = b_2 - a_1 \beta_1 - a_2 \beta_0$$

$$\beta_3 = b_3 - a_1 \beta_2 - a_2 \beta_1 - a_3 \beta_0$$

$$\vdots$$

$$\beta_n = b_n - a_1 \beta_{n-1} - \cdots - a_{n-1} \beta_1 - a_n \beta_0$$

with the present choice of state variables we obtain

$$\dot{x}_1 = x_2 + \beta_1 u$$

$$\dot{x}_2 = x_3 + \beta_2 u$$

$$\vdots$$

$$\dot{x}_{n-1} = x_n + \beta_{n-1} u$$

$$\dot{x}_n = -a_n x_1 - a_{n-1} x_2 - \cdots - a_1 x_n + \beta_n u$$

In term of vector-matrix equations.

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \vdots\\ \dot{x}_{n-1}\\ \dot{x}_n\end{bmatrix} = \begin{bmatrix}0&1&0&\cdots&0\\0&0&1&\cdots&0\\ \vdots&\vdots&\vdots&&\vdots\\0&0&0&&1\\-a_n&-a_{n-1}&-a_{n-2}&&-a_1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\ \vdots\\x_{n-1}\\x_n\end{bmatrix}$$

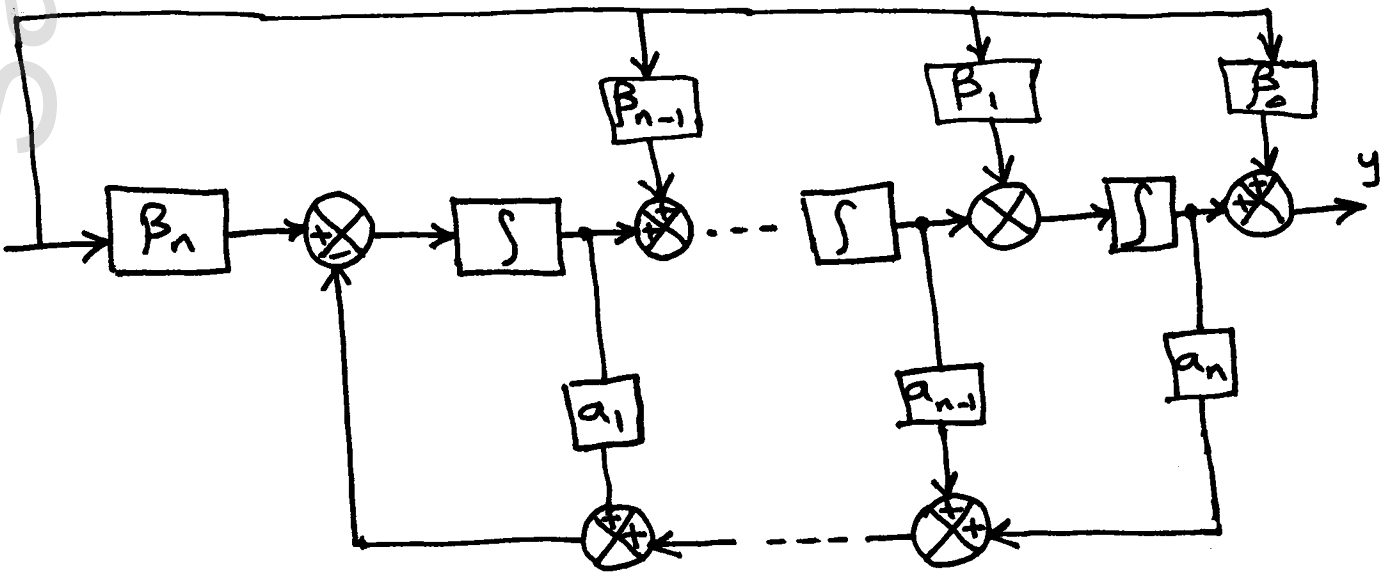
$$+ \underbrace{\begin{bmatrix}\beta_1\\ \beta_2\\ \vdots\\ \beta_{n-1}\\ \beta_n\end{bmatrix}}_{B} u$$

$$y = \underbrace{\begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix}}_{C}\begin{bmatrix}x_1\\x_2\\ \vdots\\x_n\end{bmatrix} + \underbrace{\beta_0}_{D}\, u$$

$$\dot{x} = AX + Bu, \quad y = CX + Du$$

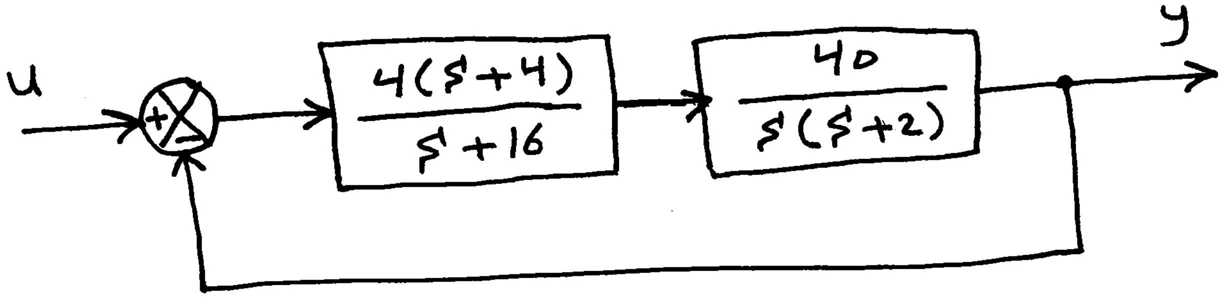
Note that the state - space representation for the T.F

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$



Block diagram realization of state equation and output equation

Example :- Consider the control system shown below



The close-loop transfer function is

$$\frac{Y(s)}{U(s)} = \frac{160\,(s+4)}{s^3 + 18s^2 + 192s + 640}$$

The corresponding differential equation is :-

$$\dddot{y} + 18\ddot{y} + 192\dot{y} + 640y = 160\dot{u} + 640u$$

$$X_1 = y - \beta_0 u$$

$$X_2 = \dot{y} - \beta_0 \dot{u} - \beta_1 u = \dot{X_1} - \beta_1 u$$

$$X_3 = \ddot{y} - \beta_0 \ddot{u} - \beta_1 \dot{u} - \beta_2 u = \dot{X_2} - \beta_2 u$$

where β_0, β_1 & β_2 are determine as follows:-

$$\beta_0 = b_0 = 0$$

$$\beta_1 = b_1 - a_1 \beta_0 = 0$$

$$\beta_2 = b_2 - a_1 \beta_1 - a_2 \beta_0 = 160$$

$$\beta_3 = b_3 - a_1 \beta_2 - a_2 \beta_1 - a_3 \beta_0 = -2240$$

$$x_1 = y$$

$$x_2 = \dot{y}$$

$$x_3 = \ddot{y} - 160u$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3 + 160u$$

$$\dot{x}_3 = -640x_1 - 192x_2 - 18x_3 - 2240u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -640 & -192 & -18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 160 \\ 2240 \end{bmatrix} [u]$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$