Frequency Response of Linear
Time-Invariant Systems

In this section we develop the characterization of linear time-invariant systems in the frequency domain. The basic excitation signals in this development are the complex exponentials and sinusoidal functions. The characteristics of the system are described by a function of the frequency variable *co* called the frequency response, which is the Fourier transform of the impulse response *h (n)* of the system. The frequency response function completely characterizes a linear time invariant system in the frequency domain.

- **Response to Complex Exponential and Sinusoidal Signals: The Frequency Response Function**

$$\mathbf{y(n)} = \sum_{\mathbf{k=-\infty}}^{\infty} \mathbf{h(k)x(n-k)}$$

In this input –output relationship, the system is characterized in the time domain by its unit sample response $\{h(n), -\infty < n < \infty\}$

To develop a frequency-domain characterization of system, let us excite the system with the complex exponential

$$\boldsymbol{x(n) = Ae^{jwn}}, \quad -\infty < \boldsymbol{n} < \infty$$

By substituting x(n) in to y (n) we obtain the response

$$\boldsymbol{y(n)} = \sum_{\boldsymbol{k=-\infty}}^{\infty} \boldsymbol{h(k)[Ae^{jw(n-k)}]}$$

$$= \boldsymbol{A}\left[\sum_{\boldsymbol{k=-\infty}}^{\infty} \boldsymbol{h(k)e^{-jwk}}\right]\boldsymbol{e^{jwn}}$$

In fact, this term is the Fourier transform of the unit sample response *h {k)* of the system. Hence we denote this function as

$$\boldsymbol{H(w) = \sum_{k=-\infty}^{\infty} h(k)e^{-jwk}}$$

With the definition in above equation, the response of the system to the complex exponential will be

$$\boldsymbol{y(n) = AH(w)e^{jwn}}$$

Example 1: Determine the output sequence of the system with impulse response

$$\boldsymbol{h(n) = \left(\frac{1}{2}\right)^n u(n)}$$

When the input is the complex exponential sequence

$$\boldsymbol{x(n) = Ae^{j\pi n/2}, \qquad -\infty < n < \infty}$$

Solution:

$$\boldsymbol{H(w) = \sum_{n=-\infty}^{\infty} h(n)e^{-jwn} = \frac{1}{1-\frac{1}{2}e^{-jw}}}$$

$$\boldsymbol{H\left(\frac{\pi}{2}\right) = \frac{1}{1+j\frac{1}{2}} = \frac{2}{\sqrt{5}}e^{-j26.6^\circ} \qquad at\ w = \pi/2}$$

And therefore the output is

$$\boldsymbol{y(n) = A\left(\frac{2}{\sqrt{5}}e^{-j26.6^\circ}\right)e^{j\pi n/2}}$$

$$\boldsymbol{y(n) = \frac{2}{\sqrt{5}}Ae^{j(\frac{\pi n}{2}-26.6)}, \qquad -\infty < n < \infty}$$

Example 2: Determine the response of the system in example 1 to input signal

$$X(n) = 10 - 5sin\frac{\pi}{2}n + 20cos\pi n, \qquad -\infty < n < \infty$$

Solution: the frequency response of the system is:

$$H(w) = \sum_{n=-\infty}^{\infty} h(n)e^{-jwn} = \frac{1}{1 - \frac{1}{2}e^{-jw}}$$

- The **first** term in the input signal is affixed signal component corresponding to w = 0

$$H(0) = \frac{1}{1 - \frac{1}{2}} = 2$$

- The **second** term in the input signal has a frequency $\frac{\pi}{2}$ at this frequency the frequency response of the system is

$$H\left(\frac{\pi}{2}\right) = \frac{2}{\sqrt{5}}e^{-j26.6^{\circ}}$$

- the third term in the input signal has a frequency $w = \pi$ at this frequency

$$H(\pi) = \frac{2}{3}$$

Hence the response of the system to x (n) is

$$y(n) = 20 - \frac{10}{\sqrt{5}}sin\left(\frac{\pi}{2}n - 26.6\right)^{\circ} + \frac{40}{3}cos\ \pi n, \qquad -\infty < n < \infty$$