Fast Fourier Transform

FFT algorithms was first developed by Tuky and Cooley in 1965.

Decimation in frequency (DIF) FFT :

$$X(K) = \sum_{n=0}^{N-1} x(n)\, W_n^{Kn} \qquad 0 \leqslant K \leqslant N-1$$

where $W_N = e^{-J 2\pi/N}$

$N = 0, 2, 4, 8, 16 \ldots$

$$X(K) = X(0) + X(1) W_N^{K} + \cdots + X(N-1) W_N^{K(N-1)}$$

$$X(K) = X(0) + X(1) W_N^{K} + \cdots + X\left(\frac{N}{2}-1\right) W_N^{K(\frac{N}{2}-1)}$$

$$+ X\left(\frac{N}{2}\right) W^{K\frac{N}{2}} + \cdots + X(N-1) W_N^{K(N-1)}$$

$$X(K) = \sum_{n=0}^{(N/2)-1} x(n) W_N^{Kn} + \sum_{n=N/2}^{N-1} x(n) W_N^{Kn}$$

$$X(K) = \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + W_N^{K(\frac{N}{2})} \sum_{n=0}^{\frac{N}{2}-1} x\left(n+\frac{N}{2}\right) W_N^{Kn}$$

$$W_N^{N/2} = e^{-J\frac{2\pi(N/2)}{N}} = e^{-J\pi} = -1$$

$$\therefore \quad X(K) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + (-1)^K x\left(n+\frac{N}{2}\right)\right] W_N^{Kn}$$

Let $K = 2m$ as an even number

$$X(2m) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + x\left(n+\frac{N}{2}\right)\right] W_N^{2nm}$$

Let $K = 2m+1$ for odd number

$$X(2m+1) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) - x\left(n + \frac{N}{2}\right)\right] W_N^n \, W_N^{2mn}$$

$$W_N^2 = e^{-j\frac{2\pi \times 2}{N}} = e^{-j\frac{2\pi}{N/2}} = W_{N/2}$$

$$X(2m) = \sum_{n=0}^{(N/2)-1} a(n)\, W_{N/2}^{mn}$$

$$X(2m+1) = \sum_{n=0}^{(N/2)-1} b_n \, W_N^n \, W_{N/2}^{mn}$$

where

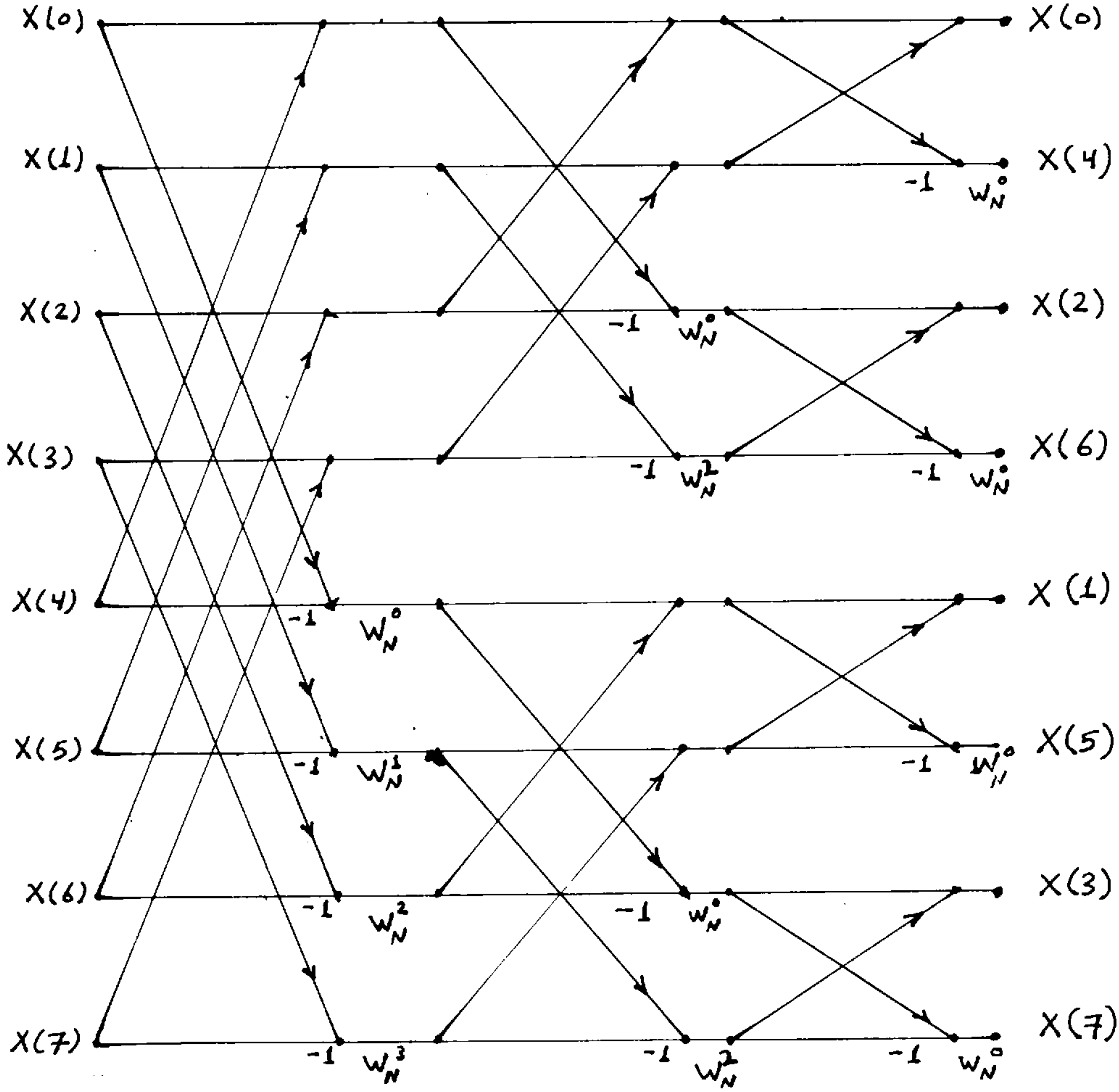
$$a(n) = x(n) + x\left(n + \frac{N}{2}\right)$$

$$N = 0, 1, \cdots \frac{N}{2} - 1$$

&

$$b(n) = x(n) - x\left(n + \frac{N}{2}\right)$$

$$N = 0, 1, \cdots, \frac{N}{2} - 1$$



eight Point DIF FFT

The Inverse FFT is defined in:-

$$x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) W_N^{-KN} = \frac{1}{N} \sum_{K=0}^{N-1} X(K) \hat{W}_N^{Kn}$$

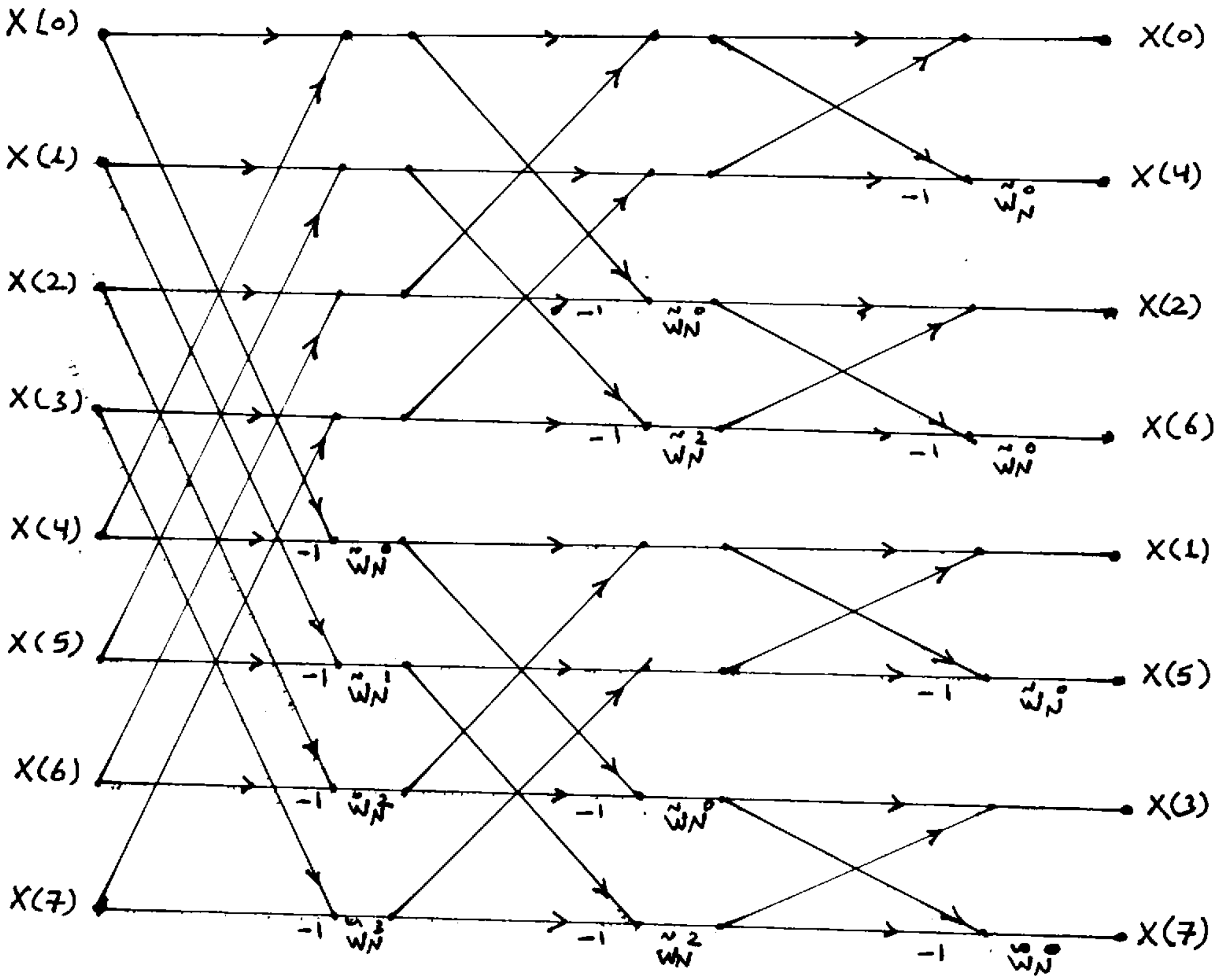
where $K = 0, 1, \dots N-1$

* Note

the W_N Factor is replaced by $\hat{W}_N$ and

the sum is multiplied by $\frac{1}{N}$ In inverse

Fast Fourier transform



eight Point DIF IFFT

Example: Given a sequence $X(n)$ for $0 \leqslant n \leqslant 3$

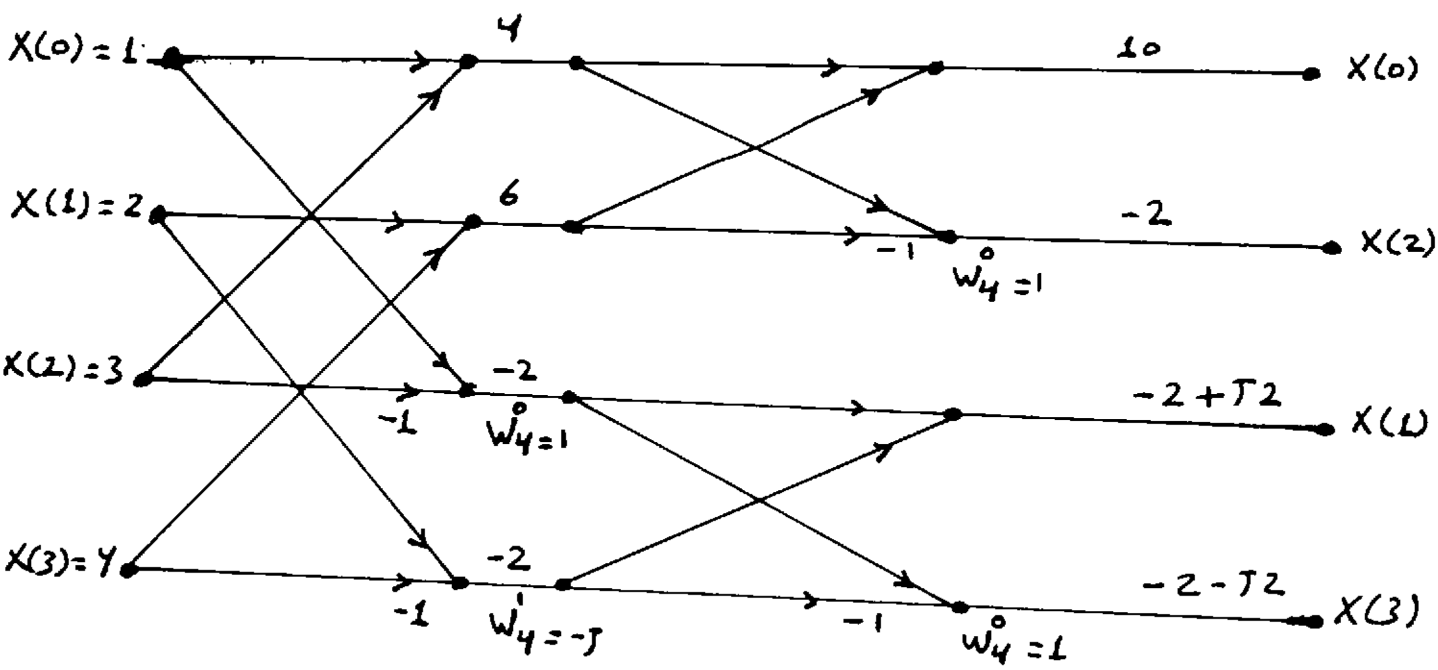
where $X(0)=1$, $X(1)=2$, $X(2)=3$, $X(3)=4$

1) Determine its DFT $X(K)$ using the decimation in frequency DIF FFT

2) Determine the no. of complex multiplication

Sol

$$W_4^0 = 1 \quad , \quad W_4^1 = -J$$



$$X(K) = [\,10 \,,\, -2+J2 \,,\, -2 \,,\, -2-J2\,]$$

b) no. of complex multiplication $= \frac{N}{2} \, Log_2(N)$

$$= \frac{4}{2} \, Log_2(4) = 4$$

Decmination in Time (DIT) FFT

in this method we split the input sequence $x(n)$ into the even indexed $x(2m)$ and $x(2m+1)$ each with N data points then

$$X(K) = \sum_{m=0}^{\frac{N}{2}-1} x(2m) W_N^{2mK} + \sum_{m=0}^{\frac{N}{2}-1} x(2m+1) W_N^{K} W_N^{2mK}$$

for $K = 0, 1, \ldots, N-1$

using $W_N^2 = W_{N/2}$

$$X(K) = \sum_{m=0}^{\frac{N}{2}-1} x(2m) W_{N/2}^{mK} + W_N^{K} \sum_{m=0}^{\frac{N}{2}-1} x(2m+1) W_{N/2}^{mK}$$

$K = 0, 1, \ldots, N-1$

$$G(K) = \sum_{m=0}^{\frac{N}{2}-1} X(2m) W_{N/2}^{mK} = \text{DFT}\left[x(2m) \text{ with } (N/2) \text{ Points}\right]$$

$$H(K) = \sum_{m=0}^{\frac{N}{2}-1} X(2m+1) W_{N/2}^{mK} = \text{DFT}\left[X(2m+1) \text{ with } (N/2) \text{ Points}\right]$$

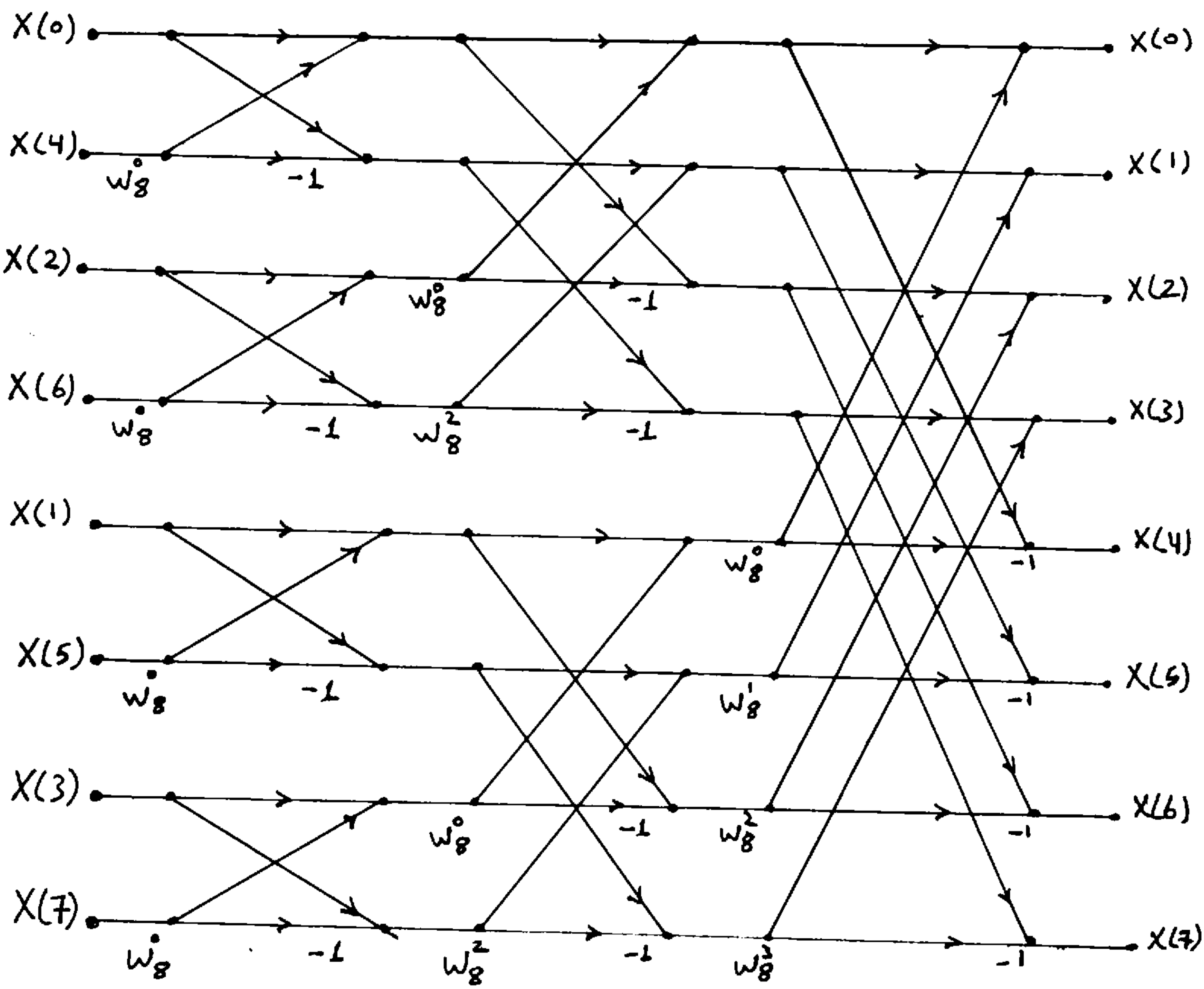
$$G(K) = G\left(K+\frac{N}{2}\right), \quad K = 0, 1, \ldots \frac{N}{2}-1$$

$$H(K) = H\left(K+\frac{N}{2}\right), \quad K = 0, 1, \ldots \frac{N}{2}-1$$

$\therefore\ X(K) = G(K) + W_N^K\, H(K)\,, \quad K = 0, 1, \ldots, \frac{N}{2}-1$

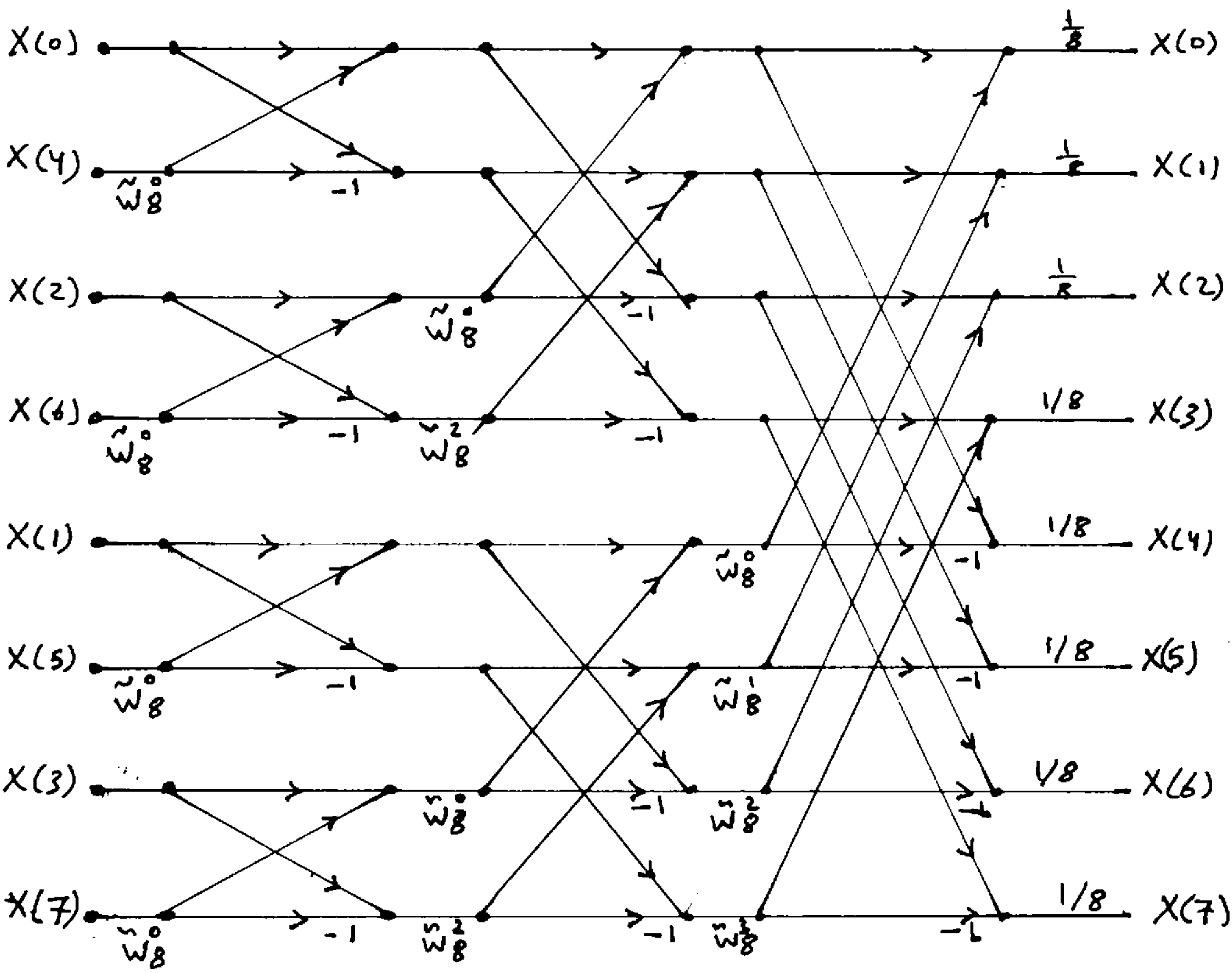
$$W_N^{\frac{N}{2}+K} = -W_N^K$$

$$X\left(\frac{N}{2}+K\right) = G(K) - W_N^K\, H(K)\,, \text{ for } K = 0, 1, \ldots \frac{N}{2}-1$$



eight Point DIT FFT

Similar to the method of Decmination in Frequency after we change W_N to $\tilde{W}_N$ and multiply the output sequence by a factor of $1/N$ we derive the inverse FFT (IFFT)



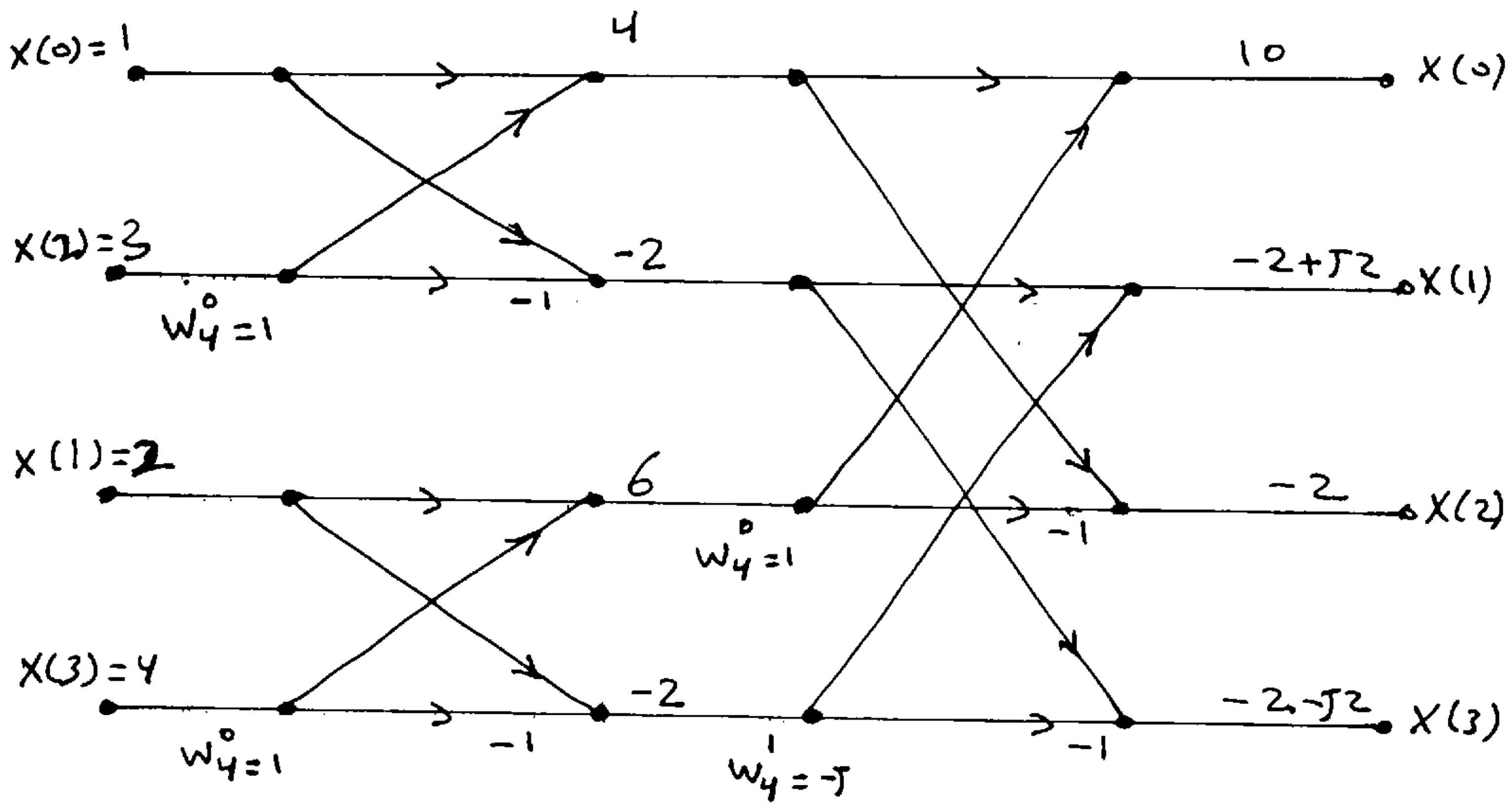
eight Point (DIT) IFFT

Example Given a sequence $x(n)$ for $0 \leq n \leq 3$ where

$x(0) = 1$, $x(1) = 2$, $x(2) = 3$ and $x(3) = 4$

Evaluate its DFT $X(K)$ using Decimation in time FFT method

Sol



H.W Find the DFT for $x(n) = [1, -1, -1, -1, 1, 1, 1, -1]$ using DIT FFT

ANS

$$[0,\ -\sqrt{2} + j(\sqrt{2}+2),\ 2 - j2,\ \sqrt{2} + j(\sqrt{2}-2),$$
$$4,\ \sqrt{2} - j(\sqrt{2}-2),\ 2 + j2,\ -\sqrt{2} - j(\sqrt{2}+2)]$$