# Inverse Z - Transform

1) Partial Fraction method
2) Long Division
3) Complex Inversion Integral (cauchy residue)

## 1- Partial fraction method

Ex Find inverse Z-transform of

$$X(Z) = \frac{Z^3 + 0.5Z^2 - 0.32Z}{(Z-0.5)(Z-0.2)^2} \qquad |Z| > 0.5$$

Sol So the region of convergence $|Z| > 0.5$
the sequence is positive time

$$X(Z) = \frac{Z(Z^2 + 0.5Z + 0.32)}{(Z-0.5)(Z-0.2)^2}$$

$$\frac{X(Z)}{Z} = \frac{Z^2 + 0.5Z + 0.32}{(Z-0.5)(Z-0.2)^2}$$

$$\frac{z^2+0.5z+0.32}{(z-0.5)(z-0.2)^2}=\frac{A}{z-0.5}+\frac{B_1}{z-0.2}+\frac{B_2}{(z-0.2)^2}$$

$$A=\frac{z^2+0.5z+0.32}{(z-0.5)(z-0.2)^2}\times(z-0.5)\Bigg|_{z=0.5}=2$$

$$B_2=\frac{z^2+0.5z+0.32}{(z-0.5)(z-0.2)^2}\times(z-0.2)^2\Bigg|_{z=0.2}=0.6$$

$$B_1=\lim_{z\to0.2}\frac{1}{1!}\frac{d}{dz}\left[\frac{z^2+0.5z+0.32}{z-0.5}\right]=-1$$

$$\therefore\ \frac{z^2+0.5z-0.32}{(z-0.5)(z-0.2)^2}=\frac{2}{z-0.5}+\frac{-1}{z-0.2}+\frac{0.6}{(z-0.2)^2}$$

$$\frac{X(z)}{z}=\frac{2}{z-0.5}+\frac{-1}{z-0.2}+\frac{0.6}{(z-0.2)^2}$$

$$X(z)=2\frac{z}{z-0.5}-\frac{z}{z-0.2}+0.6\frac{z}{(z-0.2)^2}$$

$$x(n)=2(0.5)^n-(0.2)^n+\frac{0.6}{0.2}n(0.2)^n\quad n\geq0$$

**EX** use Partial fraction method to find inverse Z- transform of

$$X(Z) = \frac{1}{3Z^2 - 4Z + 1}$$

**Sol**

$$\frac{X(Z)}{Z} = \frac{1}{Z(3Z^2 - 4Z + 1)} = \frac{1}{Z(3Z-1)(Z-1)}$$

$$\frac{1}{Z(3Z-1)(Z-1)} = \frac{A}{Z} + \frac{B}{3Z-1} + \frac{C}{Z-1}$$

$$A = 1 \quad , \quad B = \frac{-9}{2} \quad , \quad C = \frac{1}{2}$$

$$\frac{1}{Z(3Z-1)(Z-1)} = \frac{1}{Z} + \frac{-9/2}{3Z-1} + \frac{1/2}{Z-1}$$

$$\frac{X(Z)}{Z} = \frac{1}{Z} - \frac{9}{2}\frac{1}{3Z-1} + \frac{1}{2}\frac{1}{Z-1}$$

$$X(Z) = 1 - \frac{9}{2}\frac{Z}{3Z-1} + \frac{1}{2}\frac{Z}{Z-1}$$

$$X(n) = \delta(K) - \frac{3}{2}\left(\frac{1}{3}\right)^n + \frac{1}{2}(1)^n$$

**EX** Find $x(n)$ using Partial fraction method :-

$$X(Z) = \frac{1}{(1 - Z^{-1})(1 - 0.5 Z^{-1})}$$

**sol**

$$X(Z) = \frac{Z^2}{Z^2 (1 - Z^{-1})(1 - 0.5 Z^{-1})}$$

$$= \frac{Z^2}{(Z - 1)(Z - 0.5)}$$

$$\frac{X(Z)}{Z} = \frac{Z}{(Z - 1)(Z - 0.5)}$$

$$= \frac{A}{Z - 1} + \frac{B}{Z - 0.5}$$

$$A = (Z-1)\,\frac{X(Z)}{Z}\bigg|_{Z=1} = \frac{Z}{(Z-0.5)}\bigg|_{Z=1}$$

$$A = 2$$

$$B = (Z-0.5)\,\frac{X(Z)}{Z}\bigg|_{Z=0.5} = \frac{Z}{Z-1}\bigg|_{Z=0.5}$$

$$B = -1$$

$$\therefore \quad \frac{X(Z)}{Z} = \frac{2}{(Z-1)} + \frac{-1}{(Z-0.5)}$$

$$= \frac{2Z}{(Z-1)} + \frac{-Z}{(Z-0.5)}$$

$$X(n) = 2\,u(n) - (0.5)^n\,u(n)$$

EX The output $y(n)$ of a discrete time LTI system is $2(\frac{1}{3})^n u(n)$ when the input $x(n)$ is $u(n)$

a) Find the impulse response $h(n)$ of system

b) Find the output $y(n)$ when the input $x(n)$ is $(\frac{1}{2})^n u(n)$

Sol

a) $$x(n) = u(n) \longleftrightarrow X(z) = \frac{z}{z-1} \quad |z|>1$$

$$y(n) = 2\left(\frac{1}{3}\right)^n u(n) \longleftrightarrow Y(z) = \frac{2z}{z-\frac{1}{3}} \quad |z|>\frac{1}{3}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2(z-1)}{z-\frac{1}{3}} \quad |z|>\frac{1}{3}$$

using Partial fraction

$$\frac{H(z)}{z} = \frac{2(z-1)}{z(z-\frac{1}{3})} = \frac{C_1}{z} + \frac{C_2}{z-\frac{1}{3}}$$

$$C_1 = \left.\frac{2(z-1)}{z-\frac{1}{3}}\right|_{z=0} = 6 \quad , \quad C_2 = \left.\frac{2(z-1)}{z}\right|_{z=\frac{1}{3}} = -4$$

$$\therefore \quad H(Z) = 6 - 4\frac{Z}{Z-\frac{1}{3}} \qquad |Z| > \frac{1}{3}$$

Taking inverse Z- transform of

H(Z) we have:-

$$h(n) = 6\,\delta(n) - 4\left(\tfrac{1}{3}\right)^n u(n)$$

b) $x(n) = \left(\frac{1}{2}\right)^n u(n) \longleftrightarrow X(Z) = \dfrac{Z}{Z-\frac{1}{2}} \quad |Z| > \frac{1}{2}$

$$Y(Z) = X(Z)\,H(Z) = \frac{2Z(Z-1)}{(Z-\frac{1}{2})(Z-\frac{1}{3})} \qquad |Z| > \frac{1}{2}$$

using Partial fraction

$$\frac{Y(Z)}{Z} = \frac{2(Z-1)}{(Z-\frac{1}{2})(Z-\frac{1}{3})} = \frac{c_1}{Z-\frac{1}{2}} + \frac{c_2}{Z-\frac{1}{3}}$$

$$c_1 = \left.\frac{2(Z-1)}{Z-\frac{1}{3}}\right|_{Z=\frac{1}{2}} = -6$$

$$c_2 = \left.\frac{2(Z-1)}{Z-\frac{1}{2}}\right|_{Z=\frac{1}{3}} = 8$$

$$Y(z) = -6\frac{z}{z-\frac{1}{2}} + 8\frac{z}{z-\frac{1}{3}} \quad |z| > \frac{1}{2}$$

Taking inverse Z-transform

$$y(n) = \left[-6\left(\frac{1}{2}\right)^n + 8\left(\frac{1}{3}\right)^n\right]u(n)$$

## 2- Long Division

Note

* if the sequence is Positive time ($|Z| >$ something) then the numerator and denominator must be written in ascending order of $Z^{-1}$, then Perform Long Division

* if the sequence is negative time ($|Z| <$ something) then the numerator and denominator must be written in ascending order of $Z$ then Perform Long Division.

**Ex** Find inverse Z Transform for

$$H(Z) = \frac{Z}{(Z-2)(Z-3)} \qquad |Z| > 10$$

**Sol**

Since $X(Z)$ is a positive sequence $|Z| >$ something $(10)$ then the numerator and denomenator is arranged in ascending of $Z^{-1}$

$$X(Z) = \frac{Z}{Z^2 - 5Z + 6} = \frac{Z^{-1}}{1 - 5Z^{-1} + 6Z^{-2}}$$

$$\begin{array}{l}
\qquad\qquad Z^{-1} + 5Z^{-2} + 19Z^{-3} + 65Z^{-4} + 211Z^{-5} + \cdots \\
1 - 5Z^{-1} + 6Z^{-2} \;\big)\; Z^{-1} \\
\qquad\qquad \mp Z^{-1} \pm 5Z^{-2} \mp 6Z^{-3} \\
\hline
\qquad\qquad 5Z^{-2} - 6Z^{-3} \\
\qquad\qquad \mp 5Z^{-2} \pm 25Z^{-3} \mp 30Z^{-4} \\
\hline
\qquad\qquad 19Z^{-3} - 30Z^{-4} \\
\qquad\qquad \mp 19Z^{-3} \pm 95Z^{-4} \mp 114Z^{-5} \\
\hline
\qquad\qquad 65Z^{-4} - 114Z^{-5} \\
\qquad\qquad 65Z^{-4} \pm 325Z^{-5} \mp 390Z^{-6} \\
\hline
\qquad\qquad 211Z^{-5} - 390Z^{-6}
\end{array}$$

$$X(z) = z^{-1} + 5z^{-2} + 19z^{-3} + 65z^{-4} + 211z^{-5} + \cdots$$

$x(0) = 0$

$x(1) = 1$

$x(2) = 5$

$x(3) = 19$

$x(4) = 65$

$x(5) = 211$

and so on

<u>EX</u> Find inverse Z transform of

$$X(z) = \frac{z+20}{z(z+3)(z-2)} \qquad |z| > 5$$

<u>Sol</u>

Since $|z| > 5$ so it is +ve sequence

$$X(z) = \frac{z+20}{z^3+z^2-6z} \times \frac{z^{-3}}{z^{-3}}$$

$$= \frac{z^{-2}+20z^{-3}}{1+z^{-1}-6z^{-2}}$$

$$\begin{array}{r|l} & z^{-2}+19z^{-3}-13z^{-4} \\ \hline 1+z^{-1}-6z^{-2} & z^{-2}+20z^{-3} \\ & \mp z^{-2} \mp z^{-3} \pm z^{-4} \\ \hline & 19z^{-3}+6z^{-4} \\ & \mp 19z^{-3} \mp 19z^{-4} \pm 114z^{-5} \\ \hline & -13z^{-4}+114z^{-5} \\ & -13z^{-4}-13z^{-5} \\ \hline & \vdots \end{array}$$

$x(2) = 1$

$x(3) = 19$

$x(4) = -13$

$\vdots$

**Ex** Find inverse z transform of

$$X(z) = \frac{z+2}{(z+0.7)(z+0.8)(z+0.9)} \qquad |z| < 0.1$$

**Sol**

since $|z| < 0.1$ so its negative sequence

$$X(z) = \frac{2+z}{0.504 + 1.91z + 2.4z^2 + z^3}$$

$$\begin{array}{r|l}
 & 3.96 - 12.8z + 29.9z^2 \\
\hline
0.504 + 1.91z + 2.4z^2 + z^3 & 2z \\
 & \mp 2 \mp 7.5z \mp 9.5z^2 \mp 3.96z^3 \\
\hline
 & -6.5z - 9.5z^2 - 3.96z^3 \\
 & \mp 6.5z \pm 24.6z^2 \pm 30.7z^3 - 12.8z^4 \\
\hline
 & 15.1z^2 - 26.74z^3 + 12.8z^4
\end{array}$$

## 3- Complex Inversion Integral

Ex Find inverse z transform of

$$X(z) = \frac{1+\frac{1}{4}z^{-1}}{(1-\frac{1}{2}z^{-1})^2} \quad |z| > \frac{1}{2}$$

Sol

$$X(z) = \frac{1+\frac{1}{4}z^{-1}}{1-z^{-1}+\frac{1}{4}z^{-2}} \times \frac{z^2}{z^2}$$

$$= \frac{z^2+\frac{1}{4}z}{z^2-z+\frac{1}{4}}$$

$$= \frac{z^2+\frac{1}{4}z}{(z-\frac{1}{2})(z-\frac{1}{2})}$$

we have pole at $z=\frac{1}{2}$

$$x(n) = \sum \text{Residue } X(z)\, z^{n-1} \text{ at poles inside } C$$

$$= \underset{z\to\frac{1}{2}}{\text{Residue}}\, X(z)\, z^{n-1}$$

$$= \lim_{z \to \frac{1}{2}} \frac{d}{dz} \frac{\left(z^2 + \frac{1}{4} z\right) z^{n-1}}{\cancel{\left(z - \frac{1}{2}\right)^2}} \cancel{\left(z - \frac{1}{2}\right)^2}$$

$$= \lim_{z \to \frac{1}{2}} \frac{d}{dz} z^{n+1} + \frac{1}{4} z^{n}$$

$$= \lim_{z \to \frac{1}{2}} \left[ (n+1) z^{n} + \frac{1}{4} n z^{n-1} \right]$$

$$= (n+1) \left(\frac{1}{2}\right)^{n} + \frac{1}{4} n \left(\frac{1}{2}\right)^{n-1}$$

$$x(n) = n \left(\frac{1}{2}\right)^{n} + \left(\frac{1}{2}\right)^{n} + \frac{1}{2} n \left(\frac{1}{2}\right)^{n}$$

## Application of Z-transform

* **Difference equation**

if $x(k) \longleftrightarrow X(z)$ then

$$x(k+m) \longleftrightarrow z^{m} X(z) - \sum_{k=0}^{m-1} x(k)\, z^{m-k}$$

**Example** Find:-

$$Z[x(k+3)] = z^{3} X(z) - \sum_{k=0}^{3-1} x(k)\, z^{3-k}$$

$$= z^{3} X(z) - x(0)\, z^{3} - x(1)\, z^{2} - x(2)\, z$$

**Example** Find:-

$$Z[Y(k+5)] = z^{5} Y(z) - \sum_{k=0}^{5-1} Y(k)\, z^{5-k}$$

$$= z^{5} Y(z) - Y(0)\, z^{5} - Y(1)\, z^{4} - Y(2)\, z^{3} - Y(3)\, z^{2} - Y(4)\, z$$

EX Given adifference equation, find Z-transform of both sides

$$y(k+3) - 2y(k+1) - 3y(k) = x(k)$$

SoL Taking Z-transform of both side we get

$$Z[y(k+3) - 2y(k+1) - 3y(k)] = Z[x(k)]$$

$$z^3 Y(z) - Y(0)z^3 - Y(1)z^2 - Y(2)z - 2[Y(z)z - Y(0)z] - 3Y(z) = X(z)$$

$$Y(z)[z^3 - 2z - 3] - z^3 Y(0) - z^2 Y(1) - z[Y(2) - 2Y(0)] = X(z)$$

EX Give adifference equation, find Z-transform of both sides

$$y(k) - 5y(k-1) + 7y(k-2) + 13y(k-3) = x(k) - 6x(k-2) + 10x(k-3)$$

SoL

$$Y(z) - 5Y(z)z^{-1} + 7Y(z)z^{-2} + 13Y(z)z^{-3} = X(z) - 6z^{-2}X(z) + 10z^{-3}X(z)$$

$$Y(z)[1 - 5z^{-1} + 7z^{-2} + 13z^{-3}] = X(z)[1 - 6z^{-2} + 10z^{-3}]$$

$$H(z) = \frac{Y(z)}{X(z)}$$

$$H(Z) = \frac{1 - 6Z^{-2} + 10Z^{-3}}{1 - 5Z^{-1} + 7Z^{-2} + 13Z^{-3}} \times \frac{Z^{3}}{Z^{3}}$$

$$H(Z) = \frac{Z^{3} - 6Z + 10}{Z^{3} + 5Z^{2} + 7Z + 13}$$