Z – Transform

Definition of Z-T: The Z-transform is a very important tool in describing and analyzing digital systems. it also offers the techniques for digital filter design and frequency analysis of digital signals. The Z-transform of a causal sequence $x(n)$, designated by $X(Z)$ or $Z(x(n))$, is defined as:-

$$X(Z) = Z(x(n)) = \sum_{n=0}^{\infty} x(n)\, Z^{-n}$$

$$= x(0)Z^{-0} + x(1)Z^{-1} + x(2)Z^{-2} + \cdots$$

eq. (5.1)

where Z is the complex variable. Here, the summation taken from $n=0$ to $n=\infty$ is according to the fact that for most situations the digital signal $x(n)$ is the causal sequence that is $x(n)=0$ for $n \leqslant 0$. for non-causal system the summation starts at $n=-\infty$

Thus, the definition in equation (5.1) is referred to as a one side Z-transform or a unilateral transform. The region of convergence is defined based on the particular sequence $x(n)$ being applied.

The Z-transforms for common sequences are summarized below:

Z-transform convergence :

The Z-transform converges if:-

$$\sum_{n=-\infty}^{\infty} |x(n) r^{-n}| < \infty$$

so, it converges only for some values of (r)

Signal $x(n)$	Z-transform $X(Z)$	R.o.C
$\delta(n)$	1	All Z
$u(n)$	$\frac{1}{1-Z^{-1}}$	$\lvert Z\rvert > 1$
$a^n u(n)$	$\frac{1}{1-aZ^{-1}}$	$\lvert Z\rvert > \lvert a\rvert$
$n a^n u(n)$	$\frac{aZ^{-1}}{(1-aZ^{-1})^2}$	$\lvert Z\rvert > \lvert a\rvert$
$-a^n u(-n-1)$	$\frac{1}{1-aZ^{-1}}$	$\lvert Z\rvert < \lvert a\rvert$
$-n a^n u(-n-1)$	$\frac{aZ^{-1}}{(1-aZ^{-1})^2}$	$\lvert Z\rvert < \lvert a\rvert$
$(\cos W_o n)u(n)$	$\frac{1-Z^{-1}\cos W_o}{1-2Z^{-1}\cos W_o+Z^{-2}}$	$\lvert Z\rvert > 1$
$(\sin W_o n)u(n)$	$\frac{Z^{-1}\sin W_o}{1-2Z^{-1}\cos W_o+Z^{-2}}$	$\lvert Z\rvert > 1$
$(a^n \cos W_o n)u(n)$	$\frac{1-aZ^{-1}\cos W_o}{1-2aZ^{-1}\cos W_o+a^2Z^{-2}}$	$\lvert Z\rvert > \lvert a\rvert$
$(a^n \sin W_o n)u(n)$	$\frac{aZ^{-1}\sin W_o}{1-2aZ^{-1}\cos W_o+a^2Z^{-2}}$	$\lvert Z\rvert > \lvert a\rvert$

Properties of Z.T :

1- Linearity : The Z-Transform is a linear transformation, which implies

$$Z(a\,X_1(n) \pm b\,X_2(n)) = a\,X_1(Z) \pm b\,X_2(Z)$$

where a and b are constant.

2- Shifting : given $X(Z)$, the Z-transform of a sequence $x(n)$, the Z-transform of $x(n-m)$ the time shifted sequence is given by

$$Z[\,X(n-m)\,] = Z^{-m}\,X(Z)$$

3- Convolution given two seqences $X_1(n)$ and $X_2(n)$, their convolution can be determined as follows :-

$$x(n) = x_1(n) \otimes x_2(n) = \sum_{K=-\infty}^{\infty} x_1(K)\, x_2(n-K)$$

$$= \sum_{K=-\infty}^{\infty} x_1(n-K)\, x_2(K)$$

where $\otimes$ designates the linear convolution

in Z-transform domain, we have

$$X(Z) = X_1(Z) \cdot X_2(Z)$$

4. Multiplication by exponential

$$Z[a^n x(n)] = X(Z)\Big|_{Z \to \frac{Z}{a}}$$

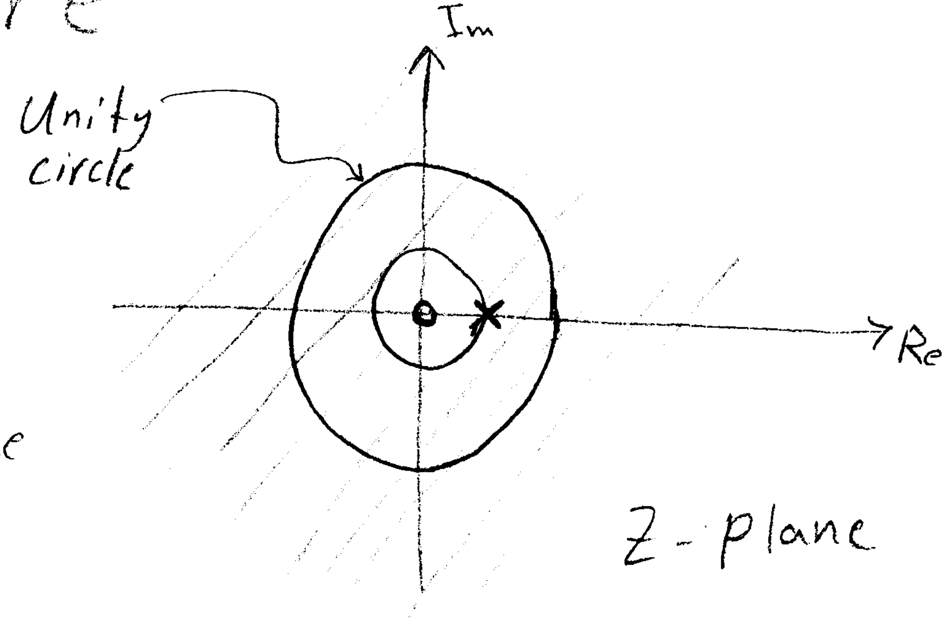
$$Z[e^{\pm an} x(n)] = X(Z)\Big|_{Z \to e^{\mp a} Z}$$

5. multiplication by n :-

$$Z[n\, x(n)] = -Z \frac{d}{dz} X(Z)$$

The Z-Plane:-

The Z-Transform X(z) convergence could be represented Graphically on the Z-Plane. Remember that Z is complex number $Z = r e^{jw}$



* always plote the Unity circle

Z-plane

* locate the Zeros of the Z-transform ⇒ roots of the numerator

* locate the poles of the Z-Transfom ⇒ the roots of the denomerator

* hilight the convergence region (regions)

Example

$$x(n) = -\left(\tfrac{1}{2}\right)^n u(-n-1)$$

$$X(Z) = \sum_{n=\infty}^{\infty} x(n)\, Z^{-n}$$

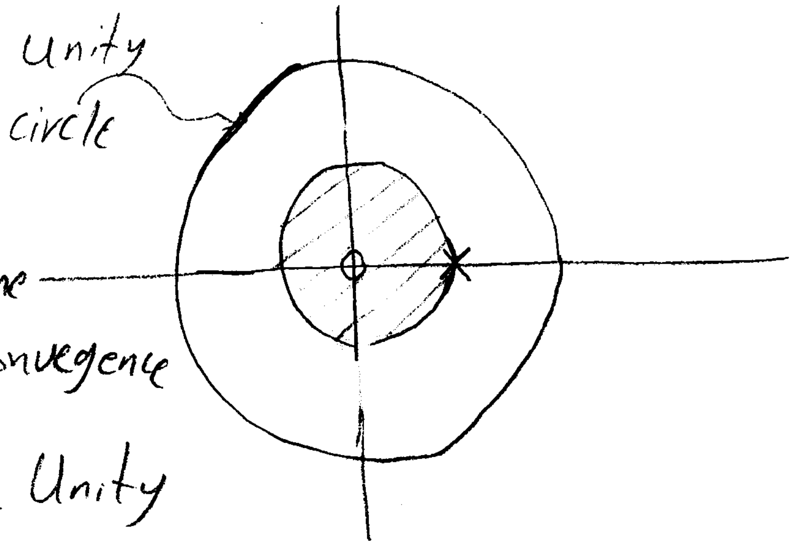
$$= \frac{1}{1-\frac{1}{2Z}} = \frac{Z}{Z-\frac{1}{2}}$$

To find the convergence reagion:-

$$\sum_{n=-\infty}^{-1} \left|\left(\tfrac{1}{2} Z^{-1}\right)^n\right| < \infty$$

it is converges if $|Z| < \frac{1}{2}$

note that the only difference between this ex. and the previous one : the area of convegence



* note that the Unity circle is outside the area of convergence so The DFT does not exist for this example

It is important to note that there **must** be no poles inside the area of convergence as the poles mean $X(Z)$ is going to (∞)

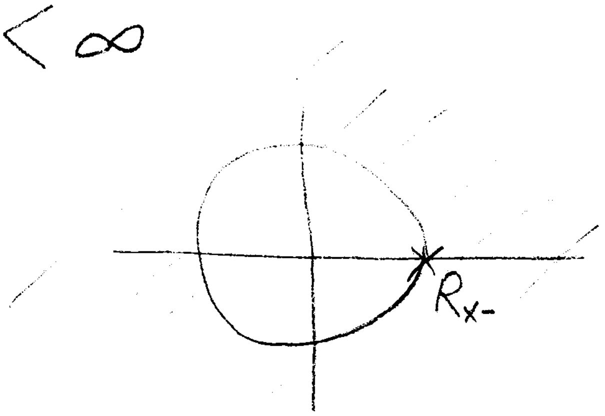
* ① The reagion of convergence is bounded by poles and the (origion or ∞)

② For a finite length sequences have area of convergence of

$$0 < |Z| < \infty$$

③ Right sided sequences ($X(n) = 0$ for $n < n_1$) have area of convergence of

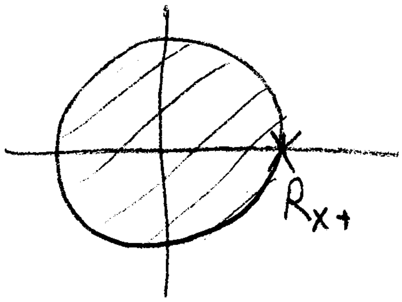
$$R_{x-} < |Z| < \infty$$



4- Left sided sequences :- $X(n) = 0$ for $n > n_1$

have areagion of convergence of

$$0 < |Z| < R_{x+}$$



5- Tow sided sequences

have a reagion of convergence of

$$R_{x-} < |Z| < R_{x+}$$

Ex :- Find Z-transform of

$$x(n) = \left(\frac{1}{2}\right)^n u(n)$$

$$X(Z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n Z^{-n}$$

$$= \sum \left(\frac{1}{2Z}\right)^n$$

$$= \frac{1}{1 - \frac{1}{2Z}}$$

$$= \frac{Z}{Z - \frac{1}{2}}$$

to test the convergence we have to look at the sum:-

$$\sum_{n=0}^{\infty} \left| \left(\frac{1}{2Z}\right)^n \right| < \infty$$

$$\therefore |Z| > \frac{1}{2}$$

EX A finite sequence $x(n)$ is defined as:

$$x(n) = [5, 3, \underset{\uparrow}{-2}, 0, 4, -3]$$

Sol

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

$$= \sum_{n=-2}^{3} x(n) Z^{-n}$$

$$= x(-2) Z^{2} + x(-1) Z + x(0) + x(2) Z^{-2} + x(3) Z^{-3}$$

$$= 5Z^{2} + 3Z - 2 + 4Z^{-2} - 3Z^{-3}$$

EX Find the Z-transform $X(Z)$ of

$$x(n) = \left(\tfrac{1}{2}\right)^n u(n) + \left(\tfrac{1}{3}\right)^n u(n)$$

Sol $\left(\tfrac{1}{2}\right)^n u(n) \leftrightarrow \dfrac{Z}{Z-\frac{1}{2}}$, $\left(\tfrac{1}{3}\right)^n u(n) \leftrightarrow \dfrac{Z}{Z-\frac{1}{3}}$

$$\therefore X(Z) = \frac{Z}{Z-\frac{1}{2}} + \frac{Z}{Z-\frac{1}{3}}$$

$$= \frac{2Z\left(Z-\frac{5}{12}\right)}{\left(Z-\frac{1}{2}\right)\left(Z-\frac{1}{3}\right)}$$

EX Find Z- transform X(Z) of

$$x(n) = \left(\tfrac{1}{3}\right)^n u(n) + \left(\tfrac{1}{2}\right)^n u(-n-1)$$

sol

$$\left(\tfrac{1}{3}\right)^n u(n) \longleftrightarrow \frac{Z}{Z-\frac{1}{3}} \qquad |Z| > \tfrac{1}{3}$$

$$\left(\tfrac{1}{2}\right)^n u(-n-1) \longleftrightarrow -\frac{Z}{Z-\frac{1}{2}} \qquad |Z| < \tfrac{1}{2}$$

$$\therefore \quad X(Z) = \frac{Z}{Z-\frac{1}{3}} - \frac{Z}{Z-\frac{1}{2}}$$

$$= \frac{1}{6} \frac{Z}{\left(Z-\frac{1}{2}\right)\left(Z-\frac{1}{3}\right)}$$

EX Find the convolution $X(n)$ of the signals below by using Z-transform

$$x_1(n) = [1, -2, 1]$$

$$X_2(n) = \begin{cases} 1 & 0 \leq n \leq 5 \\ 0 & \text{elsewhere} \end{cases}$$

SoL

$$x_1(Z) = 1 - 2Z^{-1} + Z^{-2}$$

$$X_2(Z) = 1 + Z^{-1} + Z^{-2} + Z^{-3} + Z^{-4} + Z^{-5}$$

$$X(Z) = x_1(Z)\ X_2(Z) = 1 - Z^{-1} - Z^{-6} + Z^{-7}$$

$$x(n) = [\underset{\uparrow}{1}, -1, 0, 0, 0, 0, -1, 1]$$

Saif Asaad

EX Find $Z[(n-2)\,a^{(n-2)}\cos[w(n-2)]\,u(n-2)$

Sol

$$= Z^{-2}\, Z[n\, a^{n} \cos wn\, u(n)]$$

$$= Z^{-2}\,(-Z)\,\frac{d}{dZ}\, Z\,[a^{n} \cos wn\, u(n)]$$

$$= -Z^{-1}\,\frac{d}{dZ}\,\frac{Z^2 - Z\cos w}{Z^2 - 2Z\cos w + 1}\Bigg|_{Z \to \frac{Z}{a}}$$

Saif Asaad

<u>Ex</u> Find R.O.C of the following Right-sided Exponential"

$$x[n] = a^n u(n)$$

<u>SoL</u>

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

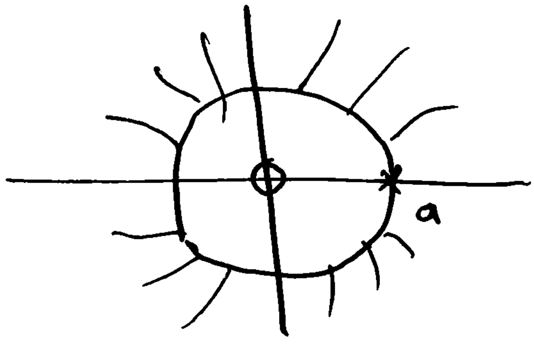
$$= \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$$

$$= \frac{1}{1 - \frac{a}{z}} = \frac{z}{z - a}$$

Zeros $z = 0$

Poles $z = a$



<u>Ex</u> Find R.O.C , Zeros , Poles "left-Sided exponential

$$X(n) = -a^n\, U(-n-1)$$

<u>SoL</u>

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$= \sum_{-\infty}^{\infty} -a^n\, u(-n-1)\, z^{-n}$$

$$= \sum_{-\infty}^{-1} -a^n\, z^{-n}$$

$$= \sum_{1}^{\infty} -a^{-n}\, z^{n}$$

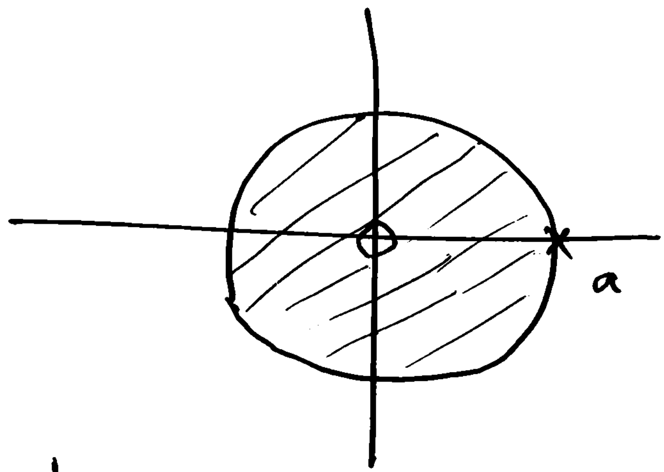
$$= \sum_{1}^{\infty} -\left(\frac{z}{a}\right)^n$$

$$= -\left(\sum_{0}^{\infty} \left(\frac{z}{a}\right)^n - 1 \right.$$

$$= -\left(\frac{1}{1-\frac{z}{a}} - 1 \right)$$

$$= \frac{-a}{a-z} + \frac{a-z}{a-z}$$

$$= \frac{-z}{a-z} = \frac{z}{z-a}$$



R.O.C

$|z| < |a|$

Notes

$$* \quad \sum_{n=0}^{\infty} \left(\frac{z}{a}\right)^n = \sum_{n=1}^{\infty} \left(\frac{z}{a}\right)^n + 1$$

$$* \quad \sum_{n=1}^{\infty} \left(\frac{z}{a}\right)^n = \sum_{n=0}^{\infty} \left(\frac{z}{a}\right)^n - 1$$

EX $\quad x(n) = \left(\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{3}\right)^n u(n)$

Sol

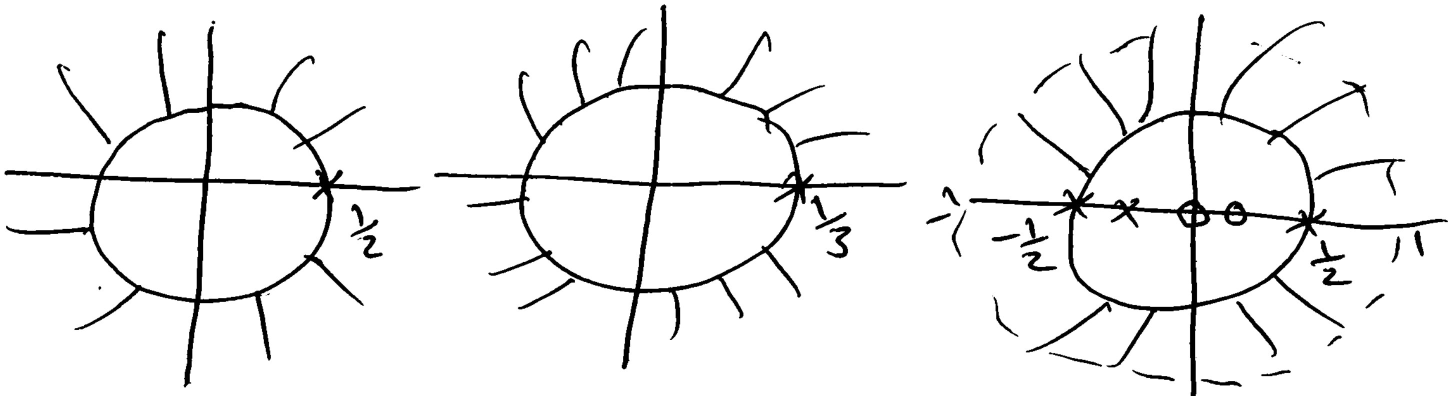
$$X(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} + \sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n z^{-n}$$

$$X(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2z}\right)^n + \sum_{n=0}^{\infty} \left(-\frac{1}{3z}\right)^n$$

$$X(z) = \frac{1}{1-\frac{1}{2z}} + \frac{1}{1+\frac{1}{3z}}$$

$$X(z) = \frac{z}{z-\frac{1}{2}} + \frac{z}{z+\frac{1}{3}}$$

$$R.o.C_T = Roc_1 \cap Roc_2$$



R.o.C.1 $\qquad$ Roc2 $\qquad$ Roc_T Stable

Pole + Zeros

$$X(z) = \frac{z}{z-\frac{1}{2}} + \frac{z}{z+\frac{1}{3}}$$

$$= \frac{z\left[\left(z+\frac{1}{3}\right)+\left(z-\frac{1}{2}\right)\right]}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)}$$

$$= \frac{z\left(2z-\frac{1}{6}\right)}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)}$$

$$= \frac{2z\left(z-\frac{1}{12}\right)}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)}$$

Poles

$z = \frac{1}{2}$, $z = -\frac{1}{3}$

Zeros

$z = 0$

$z = \frac{1}{12}$

EX $x(n) = \left(-\frac{1}{3}\right)^n u(n) - \left(\frac{1}{2}\right)^n u(-n-1)$

Sol

$$X(z) = \sum_{n=0}^{\infty} \left(\frac{-1}{3z}\right)^n - \sum_{n=-\infty}^{-1} \left(\frac{1}{2}\right)^n z^{-n}$$

$$= \sum_{0}^{\infty} \left(\frac{-1}{3z}\right)^n - \sum_{n=1}^{\infty} (2z)^n$$

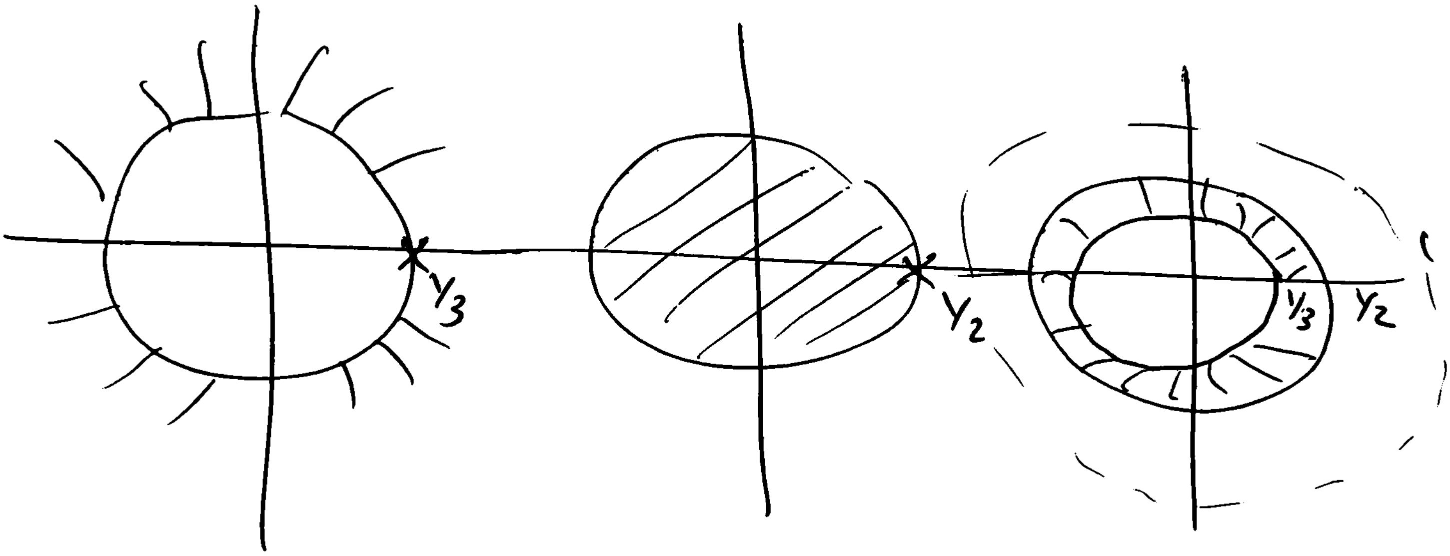
$$= \frac{1}{1+\frac{1}{3z}} - \left(\frac{1}{1-2z} - 1\right)$$

$$= \frac{z}{z+\frac{1}{3}} - \frac{1/2}{1/2-z} - \frac{1/2-z}{1/2-z}$$

$$= \frac{z}{z+\frac{1}{3}} + \frac{-1/2+1/2-z}{1/2-z}$$

$$= \frac{z}{z+1/3} + \frac{z}{z-1/2}$$

Saif Asaad



R.O.C. 1

$z > 1/3$

ROC 2

$|z| < 1/2$

ROC T

$1/3 < |z| < 1/2$

Not stable

$$X(z) = \frac{2z(z - 1/12)}{(z + 1/3)(z - 1/2)}$$

<u>Poles</u>

$z = -1/3$

$z = 1/2$

<u>Zeros</u>

$z = 0$

$z = \frac{1}{12}$

Ex Finite length exponentials

$$x(n) = \begin{cases} a^n & n \in [0, N-1] \\ 0 & \text{else} \end{cases}$$

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} \left(\frac{a}{z}\right)^n$$

$$= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n - \sum_{n=N}^{\infty} \left(\frac{a}{z}\right)^n$$

$$= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n - \left(\frac{a}{z}\right)^N \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$$

$$= \left(\frac{1}{1-\frac{a}{z}} - \frac{a^N z^{-N}}{1-\frac{a}{z}} \right) \times \frac{z^N}{z^N}$$

$$= \left(\frac{z^N - a^N}{z^N\left(1-\frac{a}{z}\right)} \right) \times \frac{z}{z}$$

$$= \frac{z\left(z^N - a^N\right)}{z^N\left(z-a\right)}$$

Zeros

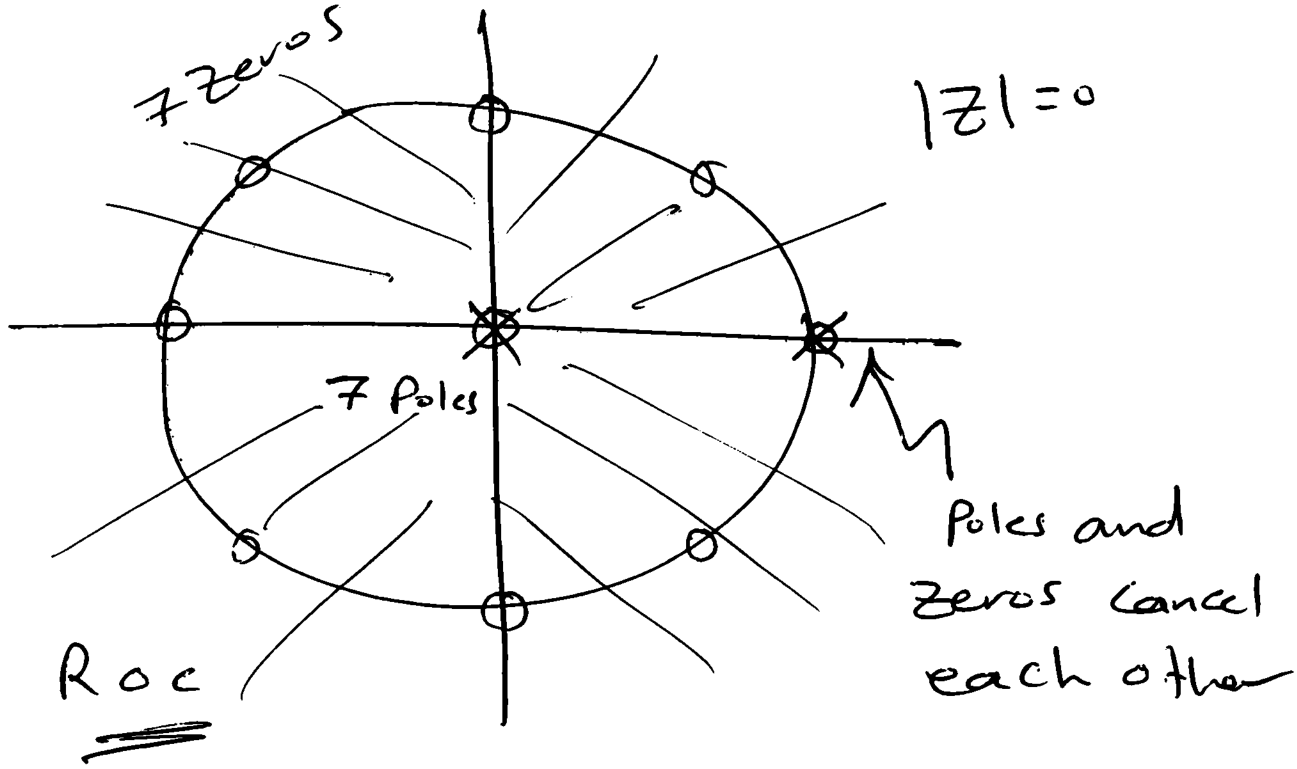
$z = 0$

$z^N = a^N$

Poles

$z^N = 0$

$z = a$



R.o.c is the whole z-Plane

except $|z| = 0$