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Spectral Density and Correlation

3.1- Energy Spectral Density.

For a real signal f(t) the energy E_f was defined as.

$$E_f = \int_{-\infty}^{\infty} f^2(t)\, dt.$$

Parseval's theorem gives a relation between ~~the~~ a time signal f(t) and it's Fourier transform F(ω)

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega. \quad \text{—— (3.1)}$$

The integral on the left-hand side of Eq 3-1 is the energy in f(t) across a one ohm resistance. Thus the quantity $|F(\omega)|^2$ is the energy per unit of frequency normalized to a resistance of one ohm.

The energy in a waveform can be found as:

$$E = \int_{-\infty}^{\infty} v(t) \cdot i^*(t)\, dt \quad \text{- - - - - (3.2)}$$

where v(t) -voltage , i(t)-current.

For resistive loads not equal to one ohm the $v(t) = Ri(t)$

Using Parseval's theorm we find the energy relations are

$$\left.\begin{aligned} E_f &= \frac{1}{R} \int_{-\infty}^{\infty} |f(t)|^2 dt \quad \text{—— (3.3a)} \\ E_f &= \frac{1}{2\pi R} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega \quad \text{—— (3.3b)} \end{aligned}\right\} \text{for } f(t) \text{ a voltage.}$$

or
$$E_f = R\int |f(t)|^2 dt \quad \text{--- 3.49}$$
for f(t) a current
$$E_f = \frac{R}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega \quad \text{--- 3.31.b}$$

$|F(\omega)|^2$ – is called the energy spectral density of the signal f(t), because the the quantity $|F(\omega)|^2$ appearing in the above expressions is in terms of energy per unit of frequency.

Q12 The energy spectral density is that function ① that describes the relative amount of energy of a given signal versus frequency, and ② whose total area under $|F(\omega)|^2$ is the energy signal.

The concept of energy spectral density (ESD) is an important one for it permits us to account for relative spectral-energy attenuation through linear systems.
let us apply a signal f(t) to the input of a linear time-invariant system whose frequency transfer function is H(ω). The spectral density of output is G(ω)

$$G(\omega) = F(\omega)H(\omega)$$

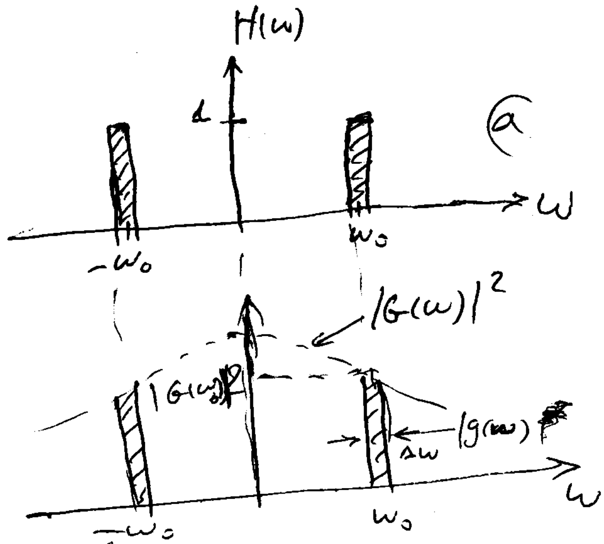
Therefore the (normalized) energy density of G(ω) is

$$|G(\omega)|^2 = |F(\omega)|^2 |H(\omega)|^2 \quad \text{---- 3.5}$$

and the normalized energy in the output signal is

$$E_g = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 \cdot |H(\omega)|^2 d\omega \quad \text{--- 3.6}$$

The useful physical interpretation of energy-density can be obtained through the use of Eq 3-6. Consider a signal $f(t)$ applied to the input of an ideal bandpass filter whose trasfer function $H(\omega)$ is shown bellow



Designating the output of the narrow-band filter as $g(t)$, We find the energy in $g(t)$ is

$$E_g = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(\omega)|^2 d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty-(\Delta\omega/2)}^{\infty+(\Delta\omega/2)} |F(\omega)|^2 |H(\omega)|^2 d\omega + \frac{1}{2\pi}\int_{\infty-(\Delta\omega/2)}^{\infty+(\Delta\omega/2)} |F(\omega)|^2 |H(\omega)|^2 d\omega$$

$$= \frac{1}{2\pi}|F(-\omega_0)|^2 \cdot \Delta\omega + \frac{1}{2\pi}|F(\omega)|^2 \Delta\omega \quad \text{--- (3-7)}$$

where $\Delta\omega$ has been assumed to be small in Eq (3-7)

For all real-valued signals the energy spectral density is an even function of ω, the half of the energy is contributed by the negative frequency components and half by the positive frequency components.

The practical significance of the above discussion can be realized by reversing the procedure. Given a pulse signal $f(t)$, how can We find it's energy spectral density?

One way is to extend the above reasoning and to construct a parallel bank of narrow-band filters, all filters positioned in frequency adjacent to each other, if we apply f(t) to this parallel bank of filters, as shown bellow we can obtain an approximation to the energy spectral distribution of f(t).

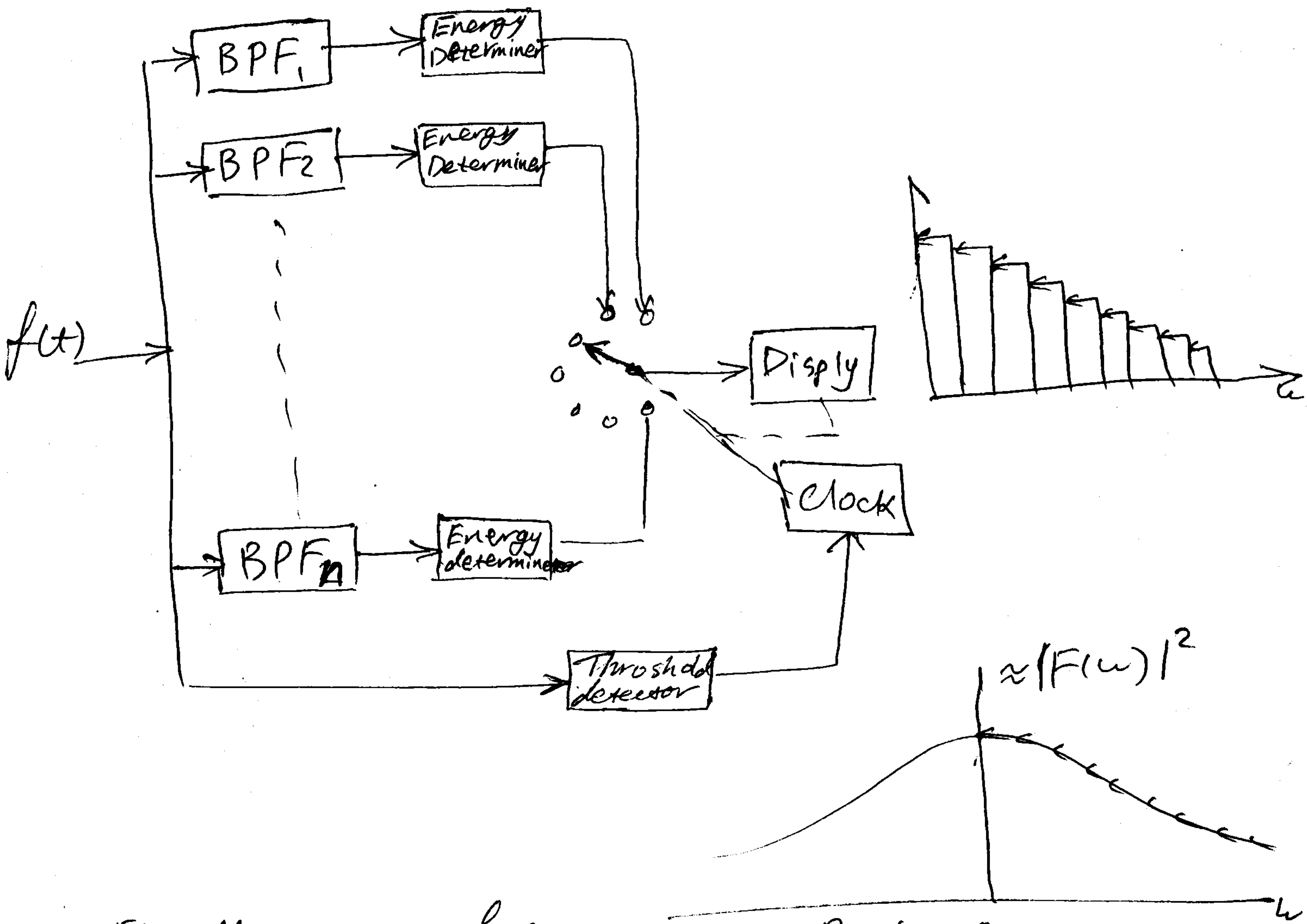


Fig. Measument of energy spectral Density

Example: A voltage signal described by $f(t) = e^{-5t}u(t)$ is applied to the input of an ideal low-pass filter. The low-frequency gain of the filter is unity, the bandwidth is 5 radians per second, and the resistance levels are 50 ohms.
Calculate the energy of the input signal and the output signal.

Solution:

The energy in the input signal $f(t)$ is

$$E_f = \frac{1}{R}\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{50}\int_{0}^{\infty} e^{-10t}\, dt$$

$$= \left(\frac{1}{50}\right)\left(\frac{1}{10}\right) = 0{,}002 \text{ joule.}$$

The energy in the output signal $g(t)$ is

$$E_g = \frac{1}{2\pi R}\int_{-\infty}^{\infty} |G(\omega)|^2 d\omega = \frac{1}{2\pi R}\int_{-\infty}^{\infty} |F(\omega)|^2 \cdot |H(\omega)|^2 d\omega$$

$$= \frac{1}{2\pi R}\int_{-5}^{5} \frac{d\omega}{\omega^2+25} \leftarrow \text{proof.}$$

$$= \frac{1}{\pi R}\int_{0}^{5} \frac{d\omega}{\omega^2+25} = \frac{1}{5\pi R}\tan^{-1}(1) = \frac{1}{5\pi R}\left(\frac{\pi}{4}\right) = 0{,}001 \text{ joule}$$

Ex: H.W. (6)

A signal $f(t) = e^{-at} u(t)$ is applied to the input of a low-pass filter with a magnitude frequency transfer function $|H(\omega)| = b/\sqrt{\omega^2+b^2}$. Determine the required relations between the constants a, b such that exactly 50% of the input signal energy, on a one-ohm basis, is transferred to the output.

3.2. Power Spectral Density.

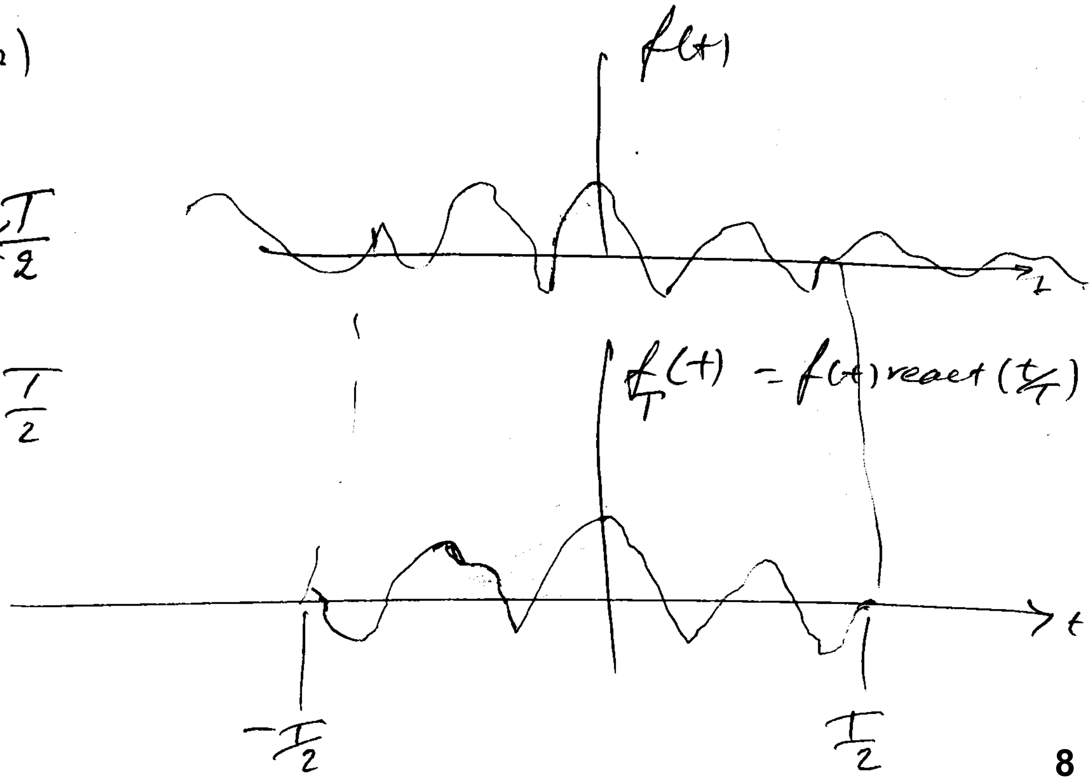
If a signal $f(t)$ exists over the entire interval $(-\infty, \infty)$, The time-averaged power of a signal (we assume a one-ohm resistance) is given by.

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |f(t)|^2 dt \quad \text{---} \quad (3\text{-}9)$$

The operation described by Eq 3-9 is the average (or mean) of the squared signal $f(t)$. Thus the quantity is called the mean-square value of the signal $f(t)$, designated simply as $\overline{f^2(t)}$.

In order to find the frequency-domain expression for the powe P, we observe that power signals have infinity energy and, therefor, may not have Fourier-trasforms, Hense in this case We consider the truncated signal $f_T(t)$ (see fig bellows)

$$f_T(t) = \begin{cases} f(t) & |t| < \frac{T}{2} \\ 0 & |t| > \frac{T}{2} \end{cases}$$



$$f_T(t) \longleftrightarrow F_T(\omega)$$

We shall assume that function $f(t)$ is finite over the finite interval $(-\frac{T}{2}, \frac{T}{2})$, Then the truncated function $f(t)\,\mathrm{rect}(t/T)$ has finite energy and its Fourier transform $F_T(\omega)$ is

$$F_T(\omega) = \mathcal{F}\{f(t)\,\mathrm{rect}(t/T)\} \quad \text{—— (3.10)}$$

Parseval's theorem for this truncated function is

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F_T(\omega)|^2 d\omega$$

Hence the average power P across a one-ohm resistor is given by

$$P = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |f(t)|^2 dt = \lim_{T\to\infty} \frac{1}{T}\frac{1}{2\pi}\int_{-\infty}^{\infty} |F_T(\omega)|^2 d\omega \quad \text{-- (3.11)}$$

The convergence of the integral on the right-hand side of Eq (3.11) permits us to interchange the order of the limiting process and integration in Eq (3.11) and we have

$$P = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{|F_T(\omega)|^2}{T} d\omega = G_f(\omega)$$

We define the power spectral density (PSD) $S_f(\omega)$ as

$$S_f(\omega) = \lim_{T\to\infty} \frac{|F_T(\omega)|^2}{T} \quad \text{------ (3.12)}$$

$$P = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_f(\omega)\, d\omega \longrightarrow S(\omega) = 2\pi \frac{dP}{d\omega} = 2\pi \frac{dG(\omega)}{d\omega} \quad \text{— (3.13)}$$

Assume that $f(t)$ is periodic and that it is represented by the exponential Fourier series.

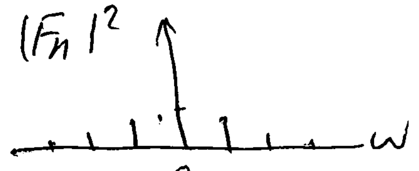
$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

Using Parseval's theorem we have

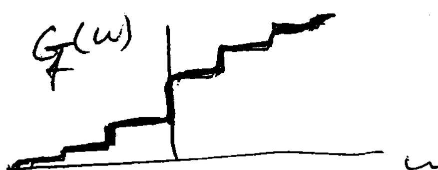
$$\overline{f^2(t)} = \sum_{n=-\infty}^{\infty} |F_n|^2 \qquad (3\text{-}134)$$

Eq (3-134) gives the power across a one-ohm sesistor at each harmonic frequency for agiven $f(t)$ and, when all terms are added, the total average power.

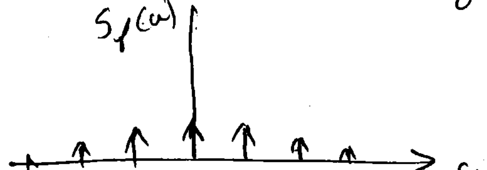
For aperiodic signal, we can use Eq (3-134) to plot a line power spectrum. as shown in Fig bellow (a). The corresponding cumulative power spectrum $G_f(\omega)$ is found simply by summing the terms in Eq (3-134) over all harmonic frequency add a discrete amount of power, this $G_f(\omega)$ will be a series of step functions forming a staircase-type graph, as shown bellow (b) (The cumulative power spectrum is always a nondecreasing function of ω because power cannot be a nagative quantity).



(a)



(b)



(c)

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Writing $G_f(\omega)$ in equation form, we have

$$G_f(\omega) = \sum_{n=-\infty}^{\infty} |F_n|^2 u(\omega - n\omega_0) \quad \text{--- } (4.15)$$

According to our definition that the derivative of step function is an impulse function Eq (3.13) and Eq (3.15) now ~~yield~~ yield the desired result

$$S_f(\omega) = 2\pi \sum_{n=-\infty}^{\infty} |F_n|^2 \delta(\omega - n\omega_0) \quad \text{--- } (4.16)$$

Q3

Therefore the power spectral density of a periodic function is a series of impulse function with weights (areas) corresponding to the magnitude squared of the respective Fourier series coefficient. this is shown in (C)

$$P = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi \sum_{n=-\infty}^{\infty} |F_n|^2 \delta(\omega - n\omega_0)\, d\omega = \sum_{n=-\infty}^{\infty} |F_n|^2$$

This result, of course, checks with Parseval's Theorem.

Example: Find the power spectral density of the periodic signal $f(t) = A\cos(\omega_0 t + \theta)$

Solution: $f(t) = \frac{A}{2} e^{j\theta} e^{j\omega_0 t} + \frac{A}{2} e^{-j\theta} e^{-j\omega_0 t}$

writing an exponential Fourier series for $f(t)$, we find $F_{-1} = \left(\frac{A}{2}\right)\exp(-j\theta)$; $F_1 = \left(\frac{A}{2}\right)\exp(j\theta)$

Using Eq. (4.16) we have

$$S_f(\omega) = \frac{1}{2}\pi A^2 \delta(\omega + \omega_0) + \frac{1}{2}\pi A^2 \delta(\omega - \omega_0)$$

$\therefore$ The mean power across one-Ohm load can be found from

$$\overline{f^2(t)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_f(\omega)\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{2}\pi A^2\,\delta(\omega+\omega_0)\,d\omega + \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{2}\pi A^2\delta(\omega-\omega_0)\,d\omega$$

$$= \frac{1}{4}A^2 + \frac{1}{4}A^2 = \frac{1}{2}A^2$$

This result can be checked easily in the time domain

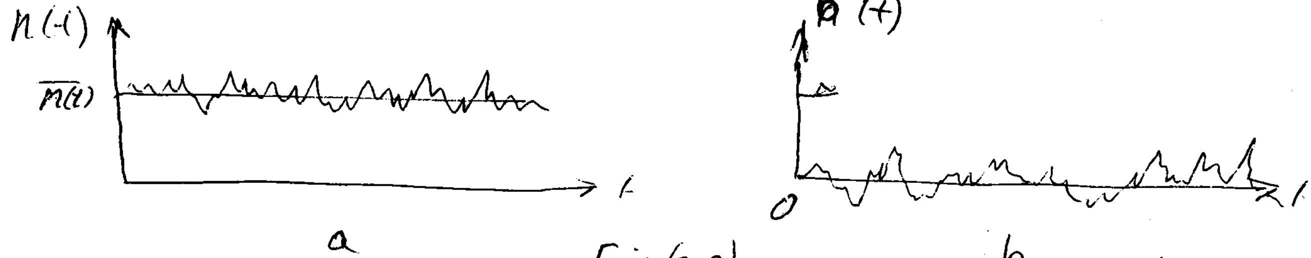
Also note that the phase information is lost in the calculation.

3.3) Time-Averaged Noise Representations

The concept of power spectra allows us to handle some of the averaged effects of the random fluctuations which are present in physical processes. These fluctuation in voltage or current tend to obscure and mask the desired signals and are commonly called noise. In ageneral sense, noise consists of any unwanted signals, random or deterministic, which interfere with the faitful reproduction of adesired signal in a systen, These unwanted signals arise from avariety of sources and can be classified as man-made or naturally occurring.

In forming averages of any signal (random or nonrandom), We find parameters which tell us something about the signal. Much of the detailed information about the signal, of course, is lost in the process. In the case of random noise, howeve, "something is better than nothing" and we will be content with averaged quantities.

Suppose $n(t)$ is a noise voltage or current (assume a one-ohm resistive load). A typical waveform is illustrated in Fig bellow



a b

Fig (3-3)

We now define the following average of $n(t)$.

1. Mean value $\overline{n(t)}$

$$\overline{n(t)} = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} n(t)\, dt$$

The parameter $\overline{n(t)}$ is often referred to as the dC, or average, value of $n(t)$.

2. Mean-square value, $\overline{n^2(t)}$

$$\overline{n^2(t)} = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |n(t)|^2\, dt \quad \text{---} \quad 3.17$$

The square root of $\overline{n^2(t)}$ is called the root-mean-square (rms) value of $n(t)$, The advantage of the rms notation is that the units of $\sqrt{\overline{n^2(t)}}$ are the same as those of $n(t)$

A side from a resistance scaling factor, Eq 4.17 gives the tim-averaged power of $n(t)$. ~~from the~~

3- AC component, $\sigma(t)$

$$\sigma(t) \triangleq n(t) - \overline{n(t)} \quad \text{———} \quad 3.18$$

The ac, or fluctuation, component of $n(t)$ is that component which remains after the mean value $\overline{n(t)}$ has been taken out; atypical wave form is shown in fig 3.3 (b).

Using Eq 3.18 in Eq 3.17 we get

$$\overline{n^2(t)} = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |\overline{n(t)} + \sigma(t)|^2\, dt.$$

$$\overline{n^2(t)} = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |\overline{n(t)}|^2 dt + \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |\sigma(t)|^2 dt \quad \text{--- (3-19)}$$

In Eq 3-19 we have used the fact that $\overline{n(t)}$ is a constant and the mean of $\sigma(t)$ is zero by definition. The term on the left-hand side of Eq 3-19 is the time-averaged power in $n(t)$ across a one-ohm resistor. likewise the first term on the RHS of Eq (3-19) is the dc power and the second term, the ac in $n(t)$.

An inspection of Eq 3-19 reveals that the rms value of $n(t)$ is equal to the rms value of $\sigma(t)$ only if the mean value $\overline{n(t)}$ is zero.

The foregoing definitions can be applied to any signal whether random or nonrandom.

Example: Calculate the (a) average value, (b) ac power, and (c) rms value of the periodic waveform $v(t) = 1 + \cos\omega_0 t$

Solution:

a) $\overline{v(t)} = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} (1+\cos\omega_0 t)\, dt = 1$

b) $\overline{\sigma^2(t)} = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} (\cos\omega_0 t)^2 dt = \frac{1}{2}$

c) $\overline{v^2(t)} = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} (1+\cos\omega_0 t)^2 dt =$

$$= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} (1 + 2\cos\omega_0 t + \cos^2\omega_0 t)\, dt = \frac{3}{2}$$

$$v_{rms} = \sqrt{\overline{v^2(t)}} = \sqrt{\frac{3}{2}}$$

An important measure of the performance of systems, particularly those involving the amplification of low-level signals, is how little noise is introduced in the system. We will use a dimensionless ratio of signal power to nois power. This ratio called the signal-to-nois ratio, can be formed by taking the ratio of the mean-square signal to the mean-square noise because the resistance factor drops.

$$S/N = \overline{s^2(t)} / \overline{n^2(t)}$$

$$[S/N]_{dB} = 10 \log_{10} \left[\overline{s^2(t)} / \overline{n^2(t)} \right]$$ – the S/N ratio in decibels.

Both $\overline{s^2(t)}$ and $\overline{n^2(t)}$ are assumed to be measured at the same point.

3-4) Correlation Functions

There is an operation in the time domain which is equivalent to finding the power spectral density in frequency.

$$S_f(\omega) = \lim_{T\to\infty} \frac{1}{T} |F(\omega)|^2 \quad \text{— PSD.} \qquad (3\text{-}20)$$

The corresponding operation in time will be the inverse Fourier transform of Eq (3-20)

$$\mathcal{F}^{-1}\{S_f(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{T\to\infty}\frac{1}{T}|F(\omega)|^2 e^{j\omega\tau}\, d\omega \qquad (3\text{-}21)$$

We have purposely chosen a new time variable τ, in Eq (3-21) because the time variable (t) is already in use the definition of $F_f(\omega)$. Interchanging the order of operations yields.

$$\mathcal{F}^{-1}\{S_f(\omega)\} = \lim_{T\to\infty}\frac{1}{2\pi T}\int_{-\infty}^{\infty} F_T^*(\omega) F_T(\omega) e^{j\omega\tau}\, d\omega$$

$$= \lim_{T\to\infty}\frac{1}{2\pi T}\int_{-\infty}^{\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} f^*(t) e^{j\omega t}\, dt \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t_1) e^{-j\omega t_1}\, dt_1\, e^{j\omega\tau}\, d\omega$$

$$= \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f^*(t)\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t_1)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega(t-t_1+\tau)}\, d\omega\right] dt_1\, dt \qquad (3\text{-}22)$$

The integration over ω within the brackets in Eq (3-22) is now recognized as $\delta(t-t_1+\tau)$, so that.

$$\mathcal{F}^{-1}\{S_f(\omega)\} = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f^*(t) \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t_1)\, \delta(t-t_1+\tau)\, dt_1\, dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f^*(t)\, f(t+\tau)\, dt. \quad \text{--- (3-23)}$$

Equation (3-23) describes the operation in the time domain that correspond to the determination of $S_f(\omega)$ in frequency. The inverse Fourier transform of $S_f(\omega)$ is called the autocorrelation function of $f(t)$ designated by $R_f(\tau)$

Then our desired result can written as.

$$R_f(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f^*(t)\, f(t+\tau)\, dt \quad \text{--- (3-24)}$$

The Fourier trasform of both side of Eq (3-23) and using Eq.(3-24) we can write

$$S_f(\omega) = \mathcal{F}\{R_f(\tau)\} \quad \text{--- (3-25)}$$

We now have another method to find the power spectral-density function, i.e first determine the autocorrelation function and then take a Fourier transform. This method is applicable to both random and deterministic signals.

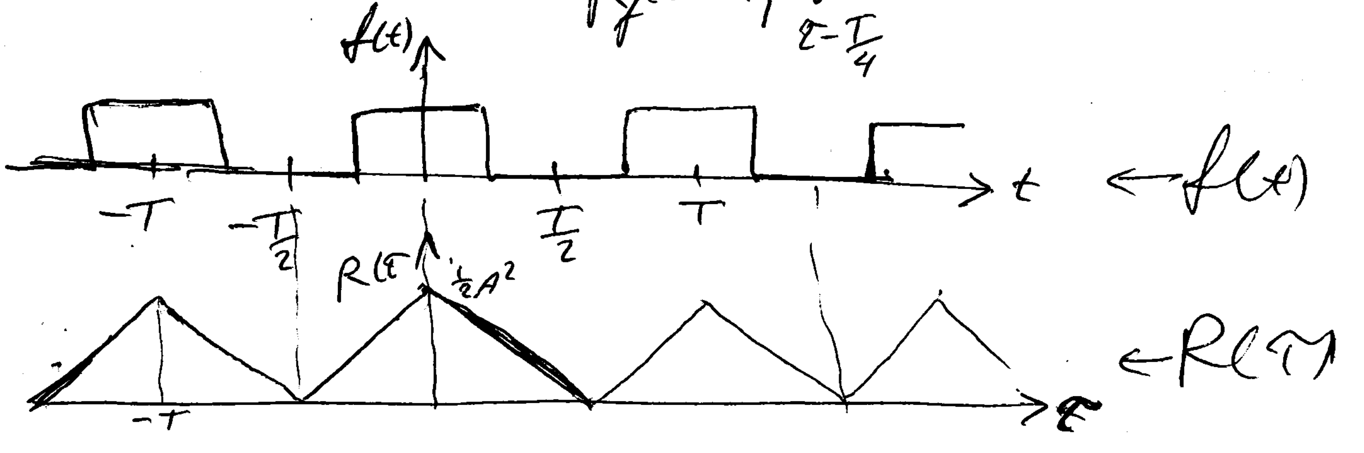
Example: Determine and sketch the autocorrelation function of aperiodic square wave with peak-to-peak amplitude A, period T, and the mean value $A/2$

Solution: Because $f(t)$ is periodic, the limiting operation in the determination of $R_f(\tau)$ can be replaced by a computation over one period. Using Eq (3-32) with this single change, we have.

For $-\frac{T}{2} < \tau < 0$

$$R_f(\tau) = \frac{1}{T}\int_{-\frac{T}{4}}^{\frac{T}{4}+\tau} A^2\,dt = A^2\left(\frac{1}{2}+\frac{\tau}{T}\right)$$

For $0 < \tau < T/2$ $\quad R_f(\tau) = \frac{1}{T}\int_{\tau-\frac{T}{4}}^{\frac{T}{4}} A^2\,dt = A^2\left(\frac{1}{2}-\frac{\tau}{T}\right).$



(19)

Example: Find the autocorrelation of $\sqrt{2}\cos(\omega_0 t+\theta)$

Solution:

$$R_f(\tau)=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} 2\cos(\omega_0 t+\theta)\cos(\omega_0 t+\omega_0\tau+\theta)\,dt$$

$$=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\cos\omega_0\tau\,dt+\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\cos(2\omega_0 t+\omega_0\tau+2\theta)\,dt$$

$$=\cos\omega_0\tau$$

Not that the autocorrelation function is independent of the phase θ.

The autocorrelation function is widely used in signal analysis. It is especially useful for the detection or recognition of signals which are masked by additive noise. for example, consider a periodic square wave such as that shown in Fig 3-4. a

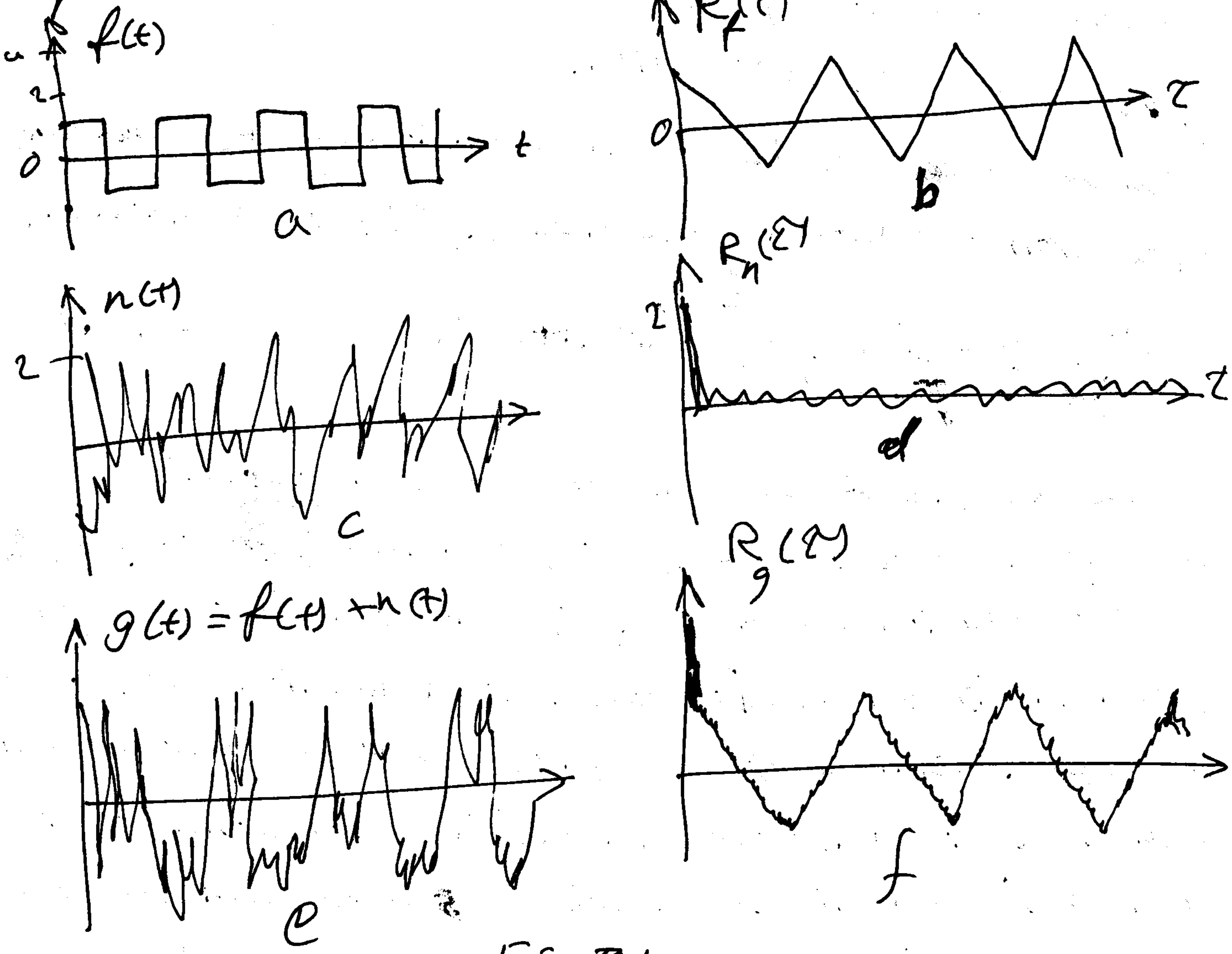


Fig 3-4

It's autocorrelation function is shown in Fig 3.4 b.

A band-limited white random-noise waveform is shown in Fig 3.4 c an It's autocorrelation function is shown in Fig 3-4. d.

The random-noise waveform is added to the periodic square wave and the resulting waveform is shown in fig 3-4. e and It's autocorrelation function is shown in fig 3-4 f.

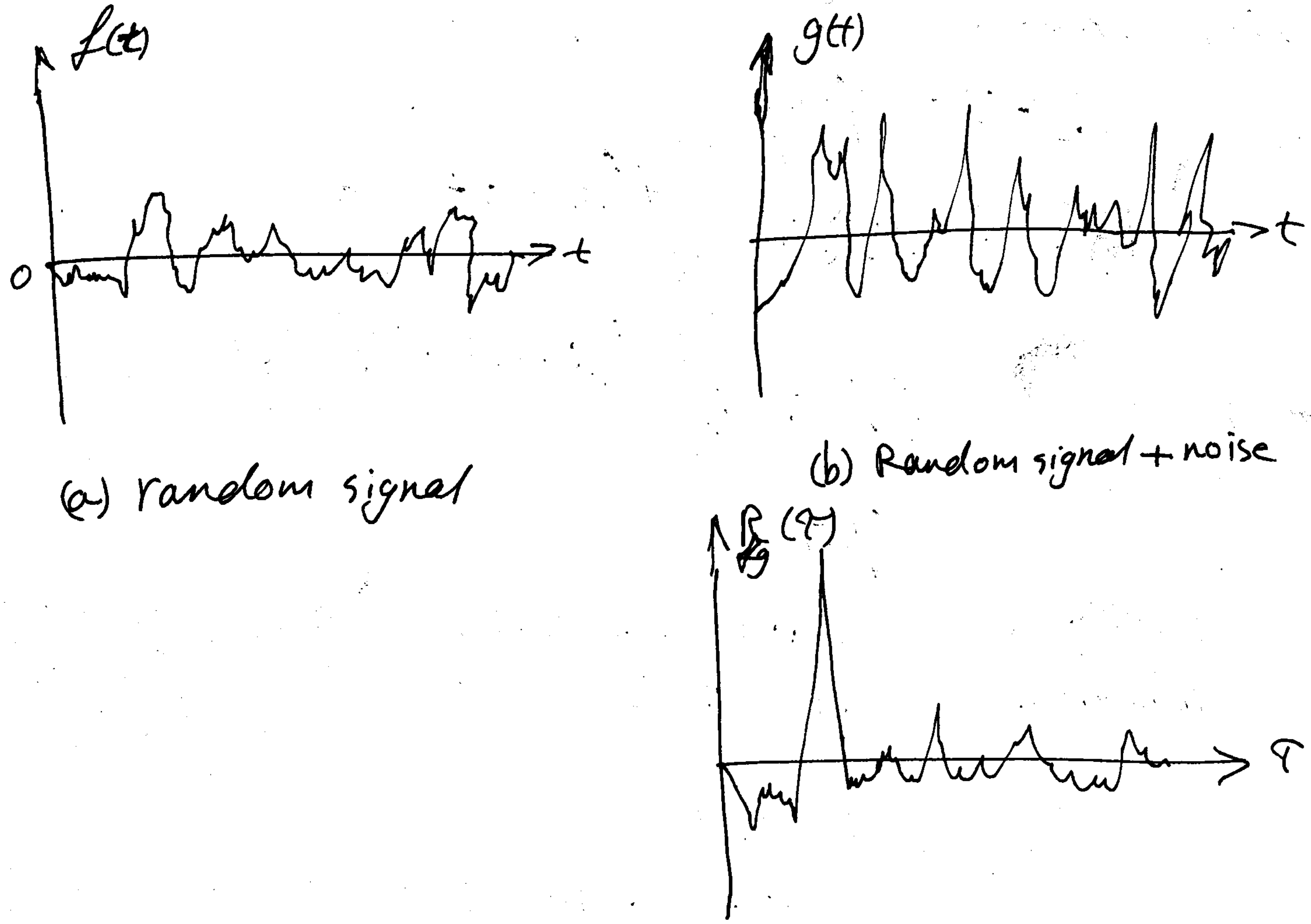
One of the drawbacks in using autocorrelation for this type of application is that the autocorrelation of noise is present in the output as well as the autocorrelation of the signal. This may make detection of aperiodic signals difficult. Also the relative time shift (phase) between signals is lost. These drawbaks can be avoided by use of a closely related operation known as crosscorrelation.

Suppos that we have two waveforms $f(t)$ and $g(t)$. The crosscorrelation function $R_{fg}(\tau)$ is defined as.

$$R_{fg}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f^*(t) g(t+\tau)\, dt. \qquad \text{—— } 3.26$$

As an example of an application of crosscorrelation, we choose a random waveform $f(t)$ as shown in Fig 3.5 a, (its autocorrelation function, $R_f(\tau)$, will be similar to that shown in Fig 3.4 d). For the second function $g(t)$, we choose a delayed replica of $f(t)$ plus a second random waveform $n(t)$ so that $g(t) = f(t-t_0) + n(t)$. The composite waveform $g(t)$ is shown in fig 3.5 b. It is presumed that the receiver has a replica of the waveform $f(t)$ available (e.g in memory). On the knowledge of $f(t)$, we wish to have the receiver make a measurment of the time delay (t_0). To do this, we take the crosscorrelation function $R_{fg}(\tau)$ as defined in Eq 3.26. The result is shown in Fig 3.5 c. The value of the time delay (t_0) is clearly evident by measuring the time delay between the origin and the large peak in the result.

Correlation function furnish measurs of the similarity of a signal $f(t)$ either with itself or with another signal versus a relative shift by an amount (τ). For dissimilar signals the peak of the correlation function is an indicator of how good this match is between signals.



(a) random signal

(b) Random signal + noise

(c) Crosscorrelation

Fig 3-5- Crosscorrelation of a random signal plus noise.

3.5) Some properties of correlation function

We have alread found, in the preceding section, that the Fourier transform of the autocrrelation function give the power spectral density. We shall examine briefly several additional properties of correlation functions.

3.5.1) Symmetry

$$R_f(-\tau) = R_f^*(\tau)$$

$$R_f(-\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f^*(t)\, f(t-\tau)\, dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f^*(\xi+\tau) f(\xi)\, d\xi$$

$$= R_f^*(\tau)$$

~~Therefore~~ Therefore the real part of $R_f(\tau)$ is an even function. and if $f(t)$ is real-valued then $S_f(-\omega) = S_f^*(\omega)$

3-5-2 Mean-Square Value

The autocorrelation function $R_f(\tau)$ evaluated at $\tau = 0$ is equal to the mean-square value of the signal $f(t)$

$$R_f(0) = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f^*(t)\, f(t)\, dt$$

$$R_f(0) = \overline{f^2(t)}$$

3.5-3) Periodicity

If $f(t+T)=f(t)$ for all t, then

$R_f(\tau+T)=R_f(\tau)$ for all τ.

3.5-4) Average Value

$$f(t)=x(t)+m_1$$

$$g(t)=y(t)+m_2$$

The crosscorrelation of $f(t)$, $g(t)$ is

$$R_{fg}(\tau)=\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}[x^*(t)+m_1][y(t+\tau)+m_2]\,dt$$

Noting that the average values of $x(t)$ and $y(t)$ are zero by definition, we have.

$$R_{fg}(\tau)=\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}x^*(t)\,y(t+\tau)\,dt+m_1m_2$$

The average value of the crosscorrelation function is

$$\overline{R_{fg}(\tau)}=\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}x^*(t)\,y(t+\tau)\,dt\,d\tau+m_1m_2$$

Interchanging the order of integration, we have

$$\overline{R_{fg}(\tau)}=\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}x^*(t)\lim\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}y(t+\tau)\,d\tau\,dt+m_1m_2$$

Because $\overline{y(t+\tau)}$ is zero, we obtain the result

$$\overline{R_{fg}(\tau)}=m_1m_2$$

Therefore the average value of the crosscorrelation of two function $f(t)$ and $g(t)$ is equal to the product of their averag values. If the average value of eithe function is zero, then the average value of their crosscorrelation is zero.

3.5.5) Maximum Value

$$|R_f(\tau)| \leq R_f(0).$$

3.56) Additivity

if two signals are added, the autocorrelation function of sum may or may not be the sum of their respective autoucorrelation functions.

$$Z(t) = X(t) + Y(t)$$

$$R_z(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} [x^*(t) + y^*(t)][x(t+\tau) + y(t+\tau)]\,dt$$

$$R_z(\tau) = R_x(\tau) + R_y(\tau) + R_{xy}(\tau) + R_{yx}(\tau).$$

We conclude that only if the crosscorrelation functions are zero [i.e if $R_{xy}(\tau) = R_{yx}(\tau) = 0$] can we write

$$R_z(\tau) = R_x(\tau) + R_y(\tau)$$

That means the average power of the sum of two signal is the sum of the average powers of the two signals only if the signals are uncorrelated. In case in which the crosscorrelation function are not zero, the signals must be added first and then the average power may be determined.

3.6) Correlation Functions for Finite-Energy Signals

The concept of correlation can be extended to signals of finite energy, Specifically, we define the autocorrelation function $r_f(\tau)$ for a signal $f(t)$ of finite energy as.

$$r_f(\tau) = \int_{-\infty}^{\infty} f^*(t)\, f(t+\tau)\, dt \quad — 3.6.1$$

Similarly, for signals $f(t)$ and $g(t)$, both of finite energy, we define the crosscorrelation function $r_{fg}(\tau)$ as.

$$r_{fg}(\tau) = \int_{-\infty}^{\infty} f^*(t)\, g(t+\tau)\, dt\,.$$

Note that for real-valued functions these operations are the same as for convolution ~~[illegible]~~

The Fourier tranform of Eq. 3-6-1 gives

$$\mathcal{F}\{r_f(\tau)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f^*(t)\, f(t+\tau)\, dt\, e^{-j\omega\tau}\, d\tau$$

Interchanging the order of integration

$$\mathcal{F}\{\gamma_f(\tau)\} = \int_{-\infty}^{\infty} f^*(t) \int_{-\infty}^{\infty} f(t+\tau)\, e^{-j\omega\tau}\, d\tau\, dt$$

Use the time-delay property of the Fourier transform in the right-hand side of the equation

We get.

$$\int_{-\infty}^{\infty} f^*(t)\, F(\omega)\, e^{-j\omega t}\, dt = |F(\omega)|^2$$

We see that:

$$\mathcal{F}\{\gamma_f(\tau)\} = |F(\omega)|^2 \quad \text{—— Eq 3-6.2}$$

Identifying the RHS of Eq 3.6.2 as the energy spectral density of $f(t)$. We conclude that the energy spectral density is the Fourier tranform of the auocorrelation function for finit-energy Signals.

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$$|G(\omega)| = \frac{\tau}{\sqrt{1+(\omega\tau/4)^2}} |Sa(\omega\tau/2)| \qquad ?$$

This system as shown in the fig. attenuates the high frequencies contained in the input spectral density and allows the lower frequencies to pass with relatively little attenuation.