كلية المنصور الجامعة

قسم
هندسة اتصالات
المرحلة الثانية

Communication I lectuers

2017 – 2018

الاتصالات

P-2

Lec.
Dr. Salman. O. M.

1500

ثاني اتصالات

اتصالات ③

The Fourier Transform and Applications

$f(t)$ is an aperiodic function as show bellow We wish to represent this function as a sum of exponential function over the entire interval $(-\infty, \infty)$ For this purpose We construct a new periodic function $f(t)$ is forced to repeat itself completely every (T) seconds.

The original function can be obtained back again by letting $T \to \infty$ that is $\lim_{T \to \infty} f_T(t) = f(t)$.

The new function $f_T(t)$ is a periodic function and consequently can be represented by an exponential

Fourie series:

$$f_T(t) = \sum_{n=-\infty}^{\infty} F_n e^{jnw_0 t} \quad \text{------------} \quad (1)$$

Where $$F_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f_T(t) e^{-jnw_0 t} \, dt \quad \text{--------} \quad (2)$$

and $$w_0 = \frac{2\pi}{T}$$

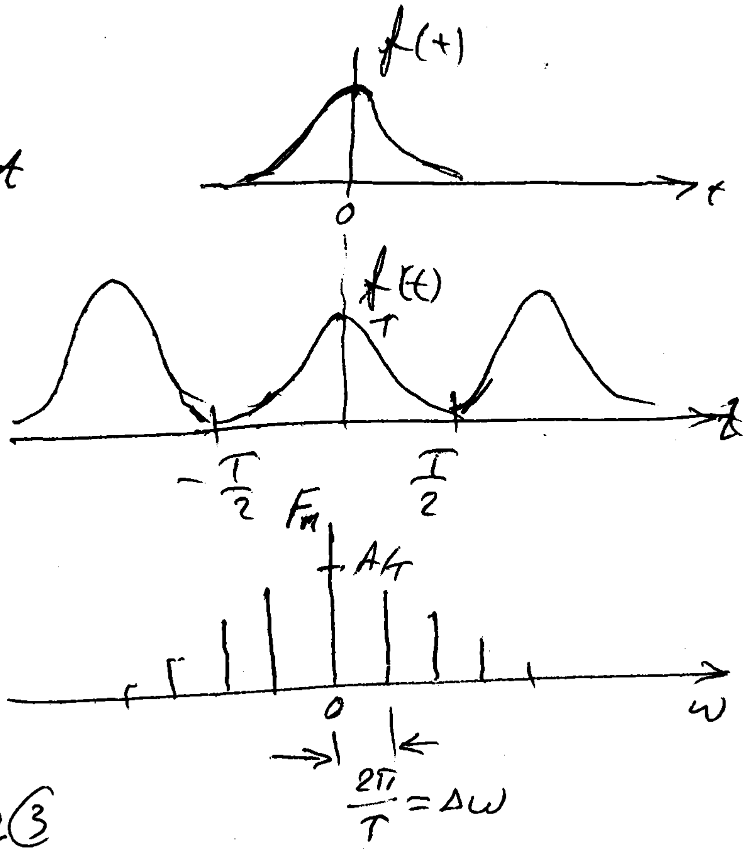
$$w_n \triangleq n w_0$$

$$F(w_n) \triangleq T F_n$$

When we use this difinitions, Eq (1), (2)

$$f_T(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T} F(\omega_n)\, e^{j\omega_n t} \quad \text{------} \; (3)$$

$$F(\omega_n) = \int_{-\frac{T}{2}}^{\frac{T}{2}} f_T(t)\, e^{-j\omega_n t}\, dt$$



The spacing between adjacent lines in the line spectrum is $\Delta\omega = \frac{2\pi}{T}$ of $f_T(t)$

Using this relation for (T) in Eq (3)

$$f_T(t) = \sum_{n=-\infty}^{\infty} F(\omega_n)\, e^{j\omega_n t} \frac{\Delta\omega}{2\pi}$$

Now as (T) becomes very large $(\Delta\omega)$ becomes small and the spectrum becomes denser, in the limit as $T \to \infty$ the discrete lines in the spectrum of $f_T(t)$ merge and the frequency spectrum becomes continuous

so: $$\lim_{T\to\infty} f_T(t) = \lim_{T\to\infty} \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} F(\omega_n)\, e^{j\omega_n t} \cdot \Delta\omega$$

becomes $$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\omega)\, e^{j\omega t}\, d\omega \quad \text{------} \; (4)$$

(2) $$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt \quad \text{------} \; (5)$$

The results expressed in Eq (4) and (5) are commonly referred to as the Fourier transform pair

Eq. (5) is known as the direct or forward Fourier tranform of $f(t)$ (more commonly, just the Fourier transform).

Eq. (4) is known as the inverse Fourier transform.

$$F(\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt \quad \text{--- (6)}$$

$$f(t) = \mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega \quad \text{--- (7)}$$

The Spectral Density. Function

Eq. (7) represents $f(t)$ as a continuous sum of exponential function with frequencies lying in the interval $(-\infty, \infty)$. The relative amplitude of the components at any frequency (ω) is proportional to $F(\omega)$. If the signal $f(t)$ ~~represent~~ a voltage. $F(\omega)$ has the dimension of voltage multiplied by time. Because frequency has the dimensions of inverse time, We can consider $F(\omega)$ as a voltage density spectrum or more generally speaking, it is known as the spectral-density function of $f(t)$.

Summarizing - a signal of finite energy can be described by a continuous spectral-density function. This spectral-density function is found by taking the Fourier transform of the signal. A periodic signal of finite average power can be described either by a set of lines on a spectral graph or by a set of impulse functions on a spectral-density graph. Each inpulse on the latter graph has an area corresponding to the height of each line, respectively, on the former graph.

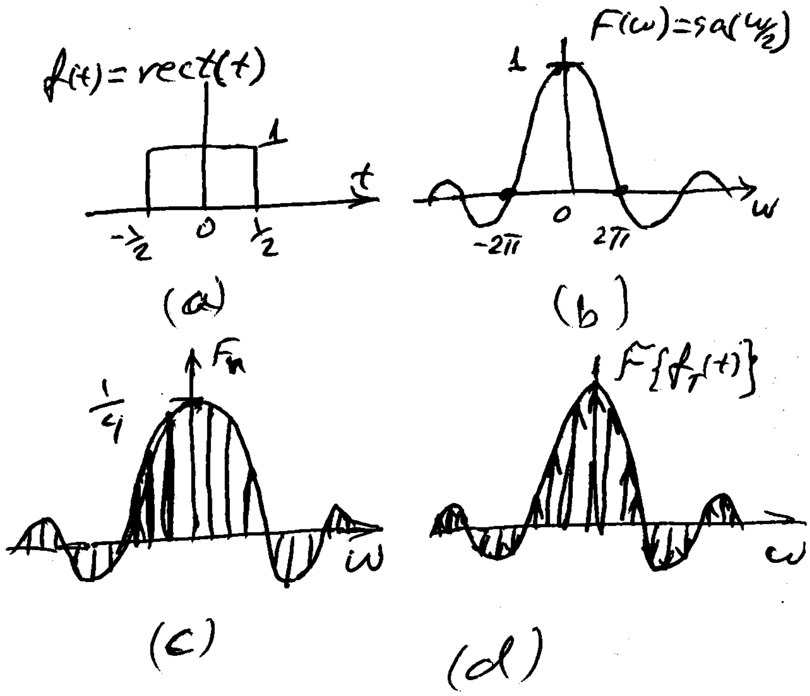
The above representation works out fine for spectral function expressed in terms of invers time (frequency)

We can write

$$\delta(2\pi f) = \frac{1}{2\pi}\delta(f) \qquad \omega = 2\pi f$$

$$\delta(f) = 2\pi\,\delta(\omega)$$

Example: find the Fourier tranform (spectral density) of the unit gate function which is shown in fig below:



Solution:

Using Eq (5) we have

$$F(\omega) = \int_{-\infty}^{\infty} rect(t)\, e^{-j\omega t}\, dt =$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j\omega t}\, dt =$$

$$= \left(e^{j\omega/2} - e^{-j\omega/2}\right)/j\omega = \frac{\sin(\omega/2)}{(\omega/2)}$$

or. $$\mathcal{F}\{rect(t)\} = Sa(\omega/2). \qquad (*)$$

Now let us determine the coefficients of the Fourier exponential series (the F_n) for this function (unit gate function) if it is repeated every 4 seconds: and plot on both a spectral and a spectral-density graph:

Solution: Using ~~Equation~~ relationship 5: $F_n = \frac{1}{T} F(\omega)\big|_{\omega = n\omega_0}$

We have

$$F_n = \frac{1}{T} F(\omega)\Big|_{\omega = n\omega_0} = \frac{1}{4} Sa(n\pi/4)$$

The line spectrum and the spectral density are shown in Fig (c) and (d)

Parseval's Theorem For Energy signals

The energy delivered to a one-ohm resistor is

$$E = \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} f(t) \cdot f^*(t)\, dt \quad ----- (8)$$

We would like to express the energy in terms of the frequency components of $f(t)$.

Using Eq (4) in Eq (8) we get

$$E = \int_{-\infty}^{\infty} f(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(\omega) e^{-j\omega t} d\omega \right] dt$$

Interchanging the order of integration on t and ω

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(\omega) \Big[\underbrace{\int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt}_{(5)} \Big] d\omega \quad -- (9)$$

But from Eq: (5) this is

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(\omega) \cdot F(\omega)\, d\omega \quad -- (10)$$

Combining Eq (8) and (10) we obtain what is known as Parseval's Theorem for Energy signals:

(6) $$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega \quad --- (11)$$

2) Some Fourier Transforms involving Impulse Function

The procedure for finding the Fourier Transform of signals of finite energy using Eq(6) is straightforward. This is not always true for signals of infinite energy. Of particular interest to us are those cases which involve the unit impulse function.

2.1. The Impulse Function

The F.T of a unit impulse, $\delta(t)$ is

$$\mathcal{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = e^{j0} = 1 \qquad (8)$$

if the impulse is time-shifted, we have

$$\mathcal{F}\{\delta(t-t_0)\} = \int_{-\infty}^{\infty} \delta(t-t_0) e^{-j\omega t} dt = e^{-j\omega t_0} \qquad (9)$$

2.2. The Eternal Complex Exponential

We could expect that the spectral density of $e^{\pm j\omega_0 t}$ will be concentrated at $\pm\omega_0$.

That this is the case is demonstrated below

$$\mathcal{F}^{-1}\{\delta(\omega \pm \omega_0)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega \pm \omega_0) e^{j\omega t} d\omega = \frac{1}{2\pi} e^{\pm j\omega_0 t} \qquad (10)$$

Taking the Fourier transform of both sides of Eq. (10) We have:

$$\mathcal{F}\,\mathcal{F}^{-1}\{\delta(\omega \pm \omega_0)\} = \frac{1}{2\pi}\,\mathcal{F}\{e^{\pm j\omega_0 t}\}$$

or by interchaning sides:

$$\mathcal{F}\{e^{\pm j\omega_0 t}\} = 2\pi\,\mathcal{F}\,\mathcal{F}^{-1}\{\delta(\omega \pm \omega_0)\}$$

$$\mathcal{F}\{e^{\pm j\omega_0 t}\} = 2\pi\,\delta(\omega \pm \omega_0) \quad \text{— (11)}$$

The spectral description of such function (complex exponential) is a line at angular frequency ω_0 (which is greater than $\omega = 0$ if rotating in a positive direction, Less than $\omega = 0$ if " " a negative " ").

In a more general sense, Eq (11) simplifies to

$$\mathcal{F}(1) = 2\pi\,\delta(\omega). \quad \text{— (12)}$$

2-3 Eternal Sinusoidal Signals

$$\mathcal{F}\{\cos\omega_0 t\} = \mathcal{F}\left\{\frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}\right\} =$$

$$= \pi\,\delta(\omega - \omega_0) + \pi\,\delta(\omega + \omega_0), \quad \text{— (13)}$$

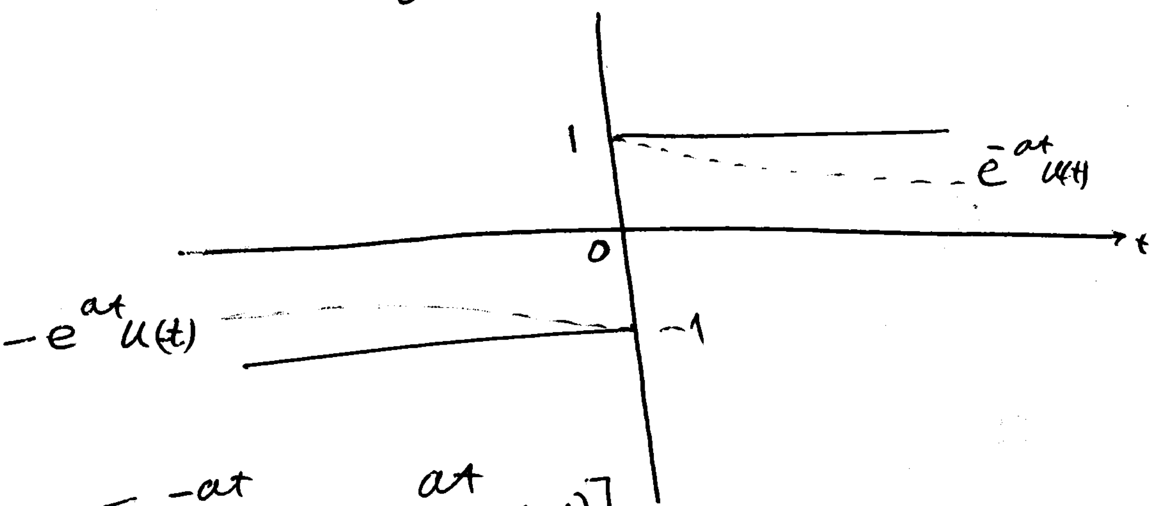
$$\mathcal{F}\{\sin\omega_0 t\} = \mathcal{F}\left\{\frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t}\right\} =$$

8 $$= [\pi\,\delta(\omega - \omega_0) - \pi\,\delta(\omega + \omega_0)]/j \quad \text{— (14)}$$

2.4. The Signum Function and the unit Step ⑨

The Signum function, $sgn(t)$, is that function which changes sign when its argument is zero.

$$sgn(t) = \frac{|t|}{t} = \begin{cases} 1 & t>0 \\ 0 & t=0 \\ -1 & t<0 \end{cases}$$



$$sgn(t) = \lim_{a \to 0} \left[e^{-at} u(t) - e^{at} u(-t) \right]$$

$$\mathcal{F}[sgn(t)] = \lim_{a\to 0} \int_0^{\infty} e^{-at} e^{-j\omega t} dt - \int_{-\infty}^{0} e^{at} e^{-j\omega t} dt$$

$$= \lim_{a\to 0} \left[\frac{1}{a+j\omega} - \frac{1}{a-j\omega} \right] = \frac{2}{j\omega} \quad \text{—— (15)}$$

The unit step function, expressed in terms of its average value and the signum function, is

$$u(t) = \frac{1}{2} + \frac{1}{2} sgn(t)$$

$$\mathcal{F}\{u(t)\} = \frac{1}{2} \mathcal{F}\{1\} + \frac{1}{2} \mathcal{F}\{sgn(t)\}.$$

Using Eq 12 and 15 we will get.

$$\mathcal{F}\{u(t)\} = \pi\,\delta(\omega) + \frac{1}{j\omega} \qquad (16)$$

2.5 Periodic Function

We can express a function f(t) which is periodic with period T by is exponential Fourier series.

$$f_T(t) = \sum_{n=-\infty}^{\infty} F_n\, e^{jn\omega_0 t} \qquad \text{where } \omega_0 = \frac{2\pi}{T}$$

Taking the Fourier transform, We find

$$\mathcal{F}\{f(t)\} = \mathcal{F}\left\{\sum_{n=-\infty}^{\infty} F_n\, e^{jn\omega_0 t}\right\}$$

If We assume the operations of integration and summation can be ~~interg~~ interchanged

$$\mathcal{F}\{f_T(t)\} = \sum_{n=-\infty}^{\infty} F_n\, \mathcal{F}\{e^{jn\omega_0 t}\}$$

Using Eq (11) we get

$$\mathcal{F}\{f_T(t)\} = 2\pi \sum_{n=-\infty}^{\infty} F_n\, \delta(\omega - n\omega_0). \qquad (17)$$

Thus the Fourier transform (spectral density) of a periodic signal consists of a set of impulses located at the harmonic frequencies of the signal.

The area (weight) of each impulse is 2π times the value of it's corresponding coefficient in the exponential Fourier series.

Example: Find the spectral-density function of an even periodic square wave whose average is zero. whose period is two seconds, and whose peak-to-peak amplitude is A

Solution: We shall use known F.T relations to obtain coefficients for the Fourier Series and then find the Fourier transform of the series.

For a rectangular pulse of unit width and height A

Eq ($*$) ~~gives~~ $\mathcal{F}\{rect(t)\} = Sa(\omega/2)$

gives

$$F(\omega) = A\,Sa(\omega/2)$$

Because $\omega_0 = \pi$ and $F_n = \frac{1}{T} F(\omega)\Big|_{\omega = n\omega_0}$, $T = 2$

$$F_n = \left(\frac{A}{2}\right) Sa(n\pi/2)$$

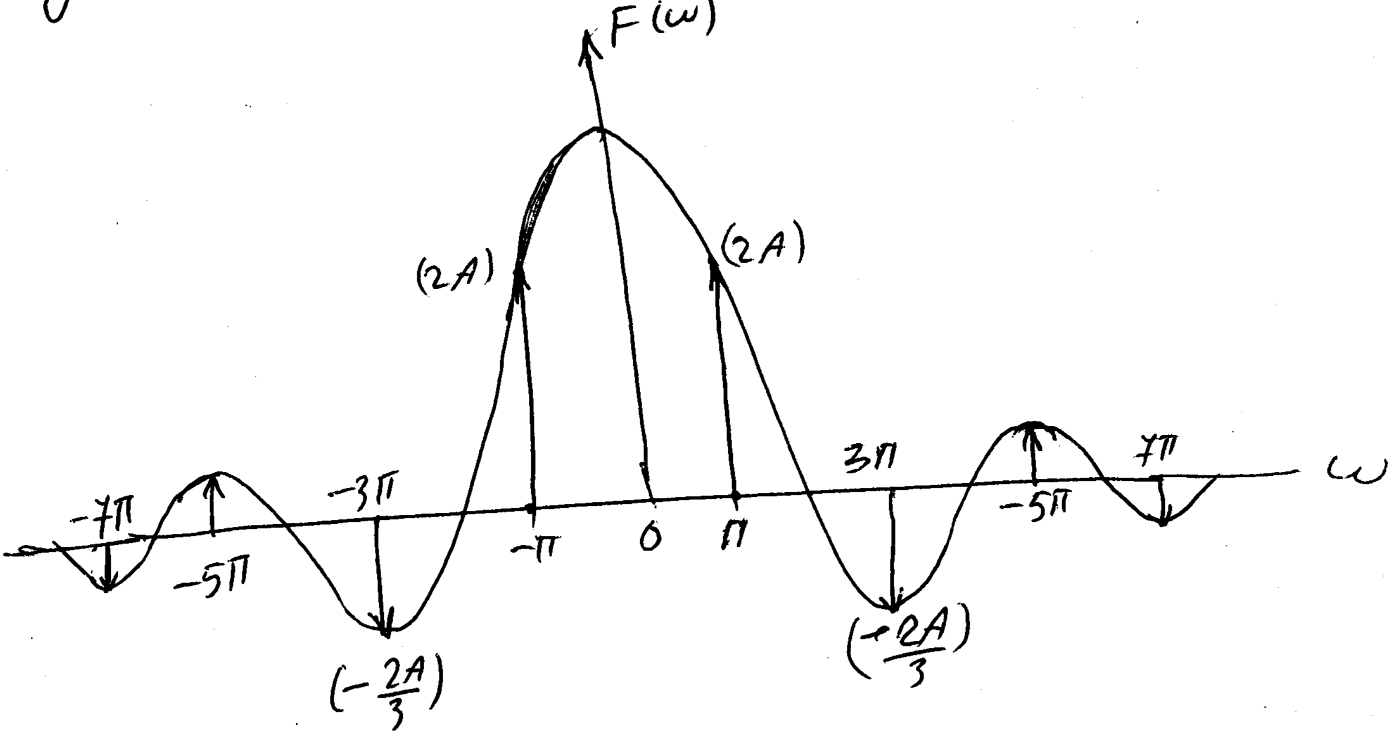
Writing the series and noting that the average value is zero, we get $f_T(t) = \frac{A}{2} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} Sa(n\pi/2) e^{jn\pi t}$

11

∴ Using Eq (17). the Fourier transform of this function is

$$\mathcal{F}\{f_T(t)\} = \pi A \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} Sa(n\pi/2)\, \delta(\omega - \pi n).$$

The spectral-density function for this example is shown bellow:
Note that the absence of an impulse at $\omega=0$ is an indicator that the average value of the signal is zero.



Properties of the Fourier Transform.

The Fourier transform is a tool for expressing a signal in terms of it's exponential components of various frequencies and is just another way of specifying the signal. We therefore have two descriptions of the same function: the time-domain and frequency-domain descriptions. It is instructive to study the effect in one domain caused by certain operations on the function in the other domain.

1. Linearity (Superposition)

$$\mathcal{F}\{a_1 f_1(t) + a_2 f_2(t)\} = a_1 F_1(\omega) + a_2 F_2(\omega)$$

or $\quad a_1 f_1(t) + a_2 f_2(t) \longleftrightarrow a_1 F_1(\omega) + a_2 F_2(\omega)$

The consequences of this property to the study of linear systems are of major importance.

2. Complex Conjugate

$$\mathcal{F}\{f^*(t)\} = F^*(-\omega)$$

proof; $\quad \mathcal{F}\{f^*(t)\} = \int_{-\infty}^{\infty} f^*(t)\, e^{-j\omega t}\, dt = \left[\int_{-\infty}^{\infty} f(t)\, e^{j\omega t}\, dt\right]^* =$

13

$$= F^*(-\omega)$$

An important consequence of this property is that if

- the signal $f(t)$ is real-valued then $f^*(t) = f(t)$ and $F^*(-\omega) = F(\omega)$.

3) Symmetry.

Any signal can be expressed as a sum of an even function $f_e(t)$ and an odd function $f_o(t)$.

$$f(t) = f_e(t) + f_o(t).$$

$$f_e = \frac{1}{2}[f(t) + f(-t)].$$ ← ?

$$f_o = \frac{1}{2}[f(t) - f(-t)].$$

$$\mathcal{F}\{f_e(t)\} = F_e(\omega) \quad \text{(and real)}$$

$$\mathcal{F}\{f_o(t)\} = F_o(\omega) \quad \text{(and imaginary)}$$

Proof: For the first part we have

$$\mathcal{F}\{f_e(t)\} = \int_{-\infty}^{\infty} f_e(t) e^{-j\omega t} dt =$$

$$= \int_{-\infty}^{\infty} f_e(t) \cos\omega t - j \int_{-\infty}^{\infty} f_e(t) \sin\omega t dt$$

$$= 2\int_{0}^{\infty} f_e(t) \cos\omega t dt.$$

Because $\cos \omega t = \cos[(-\omega)t]$ this expression is even in ω. Proof the second part follows in a similar manner.

4). Duality

A duality exists between the time domain and the frequency domain.

$$F(\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$f(t) = \mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega.$$

If

$$\mathcal{F}\{f(t)\} = F(\omega)$$

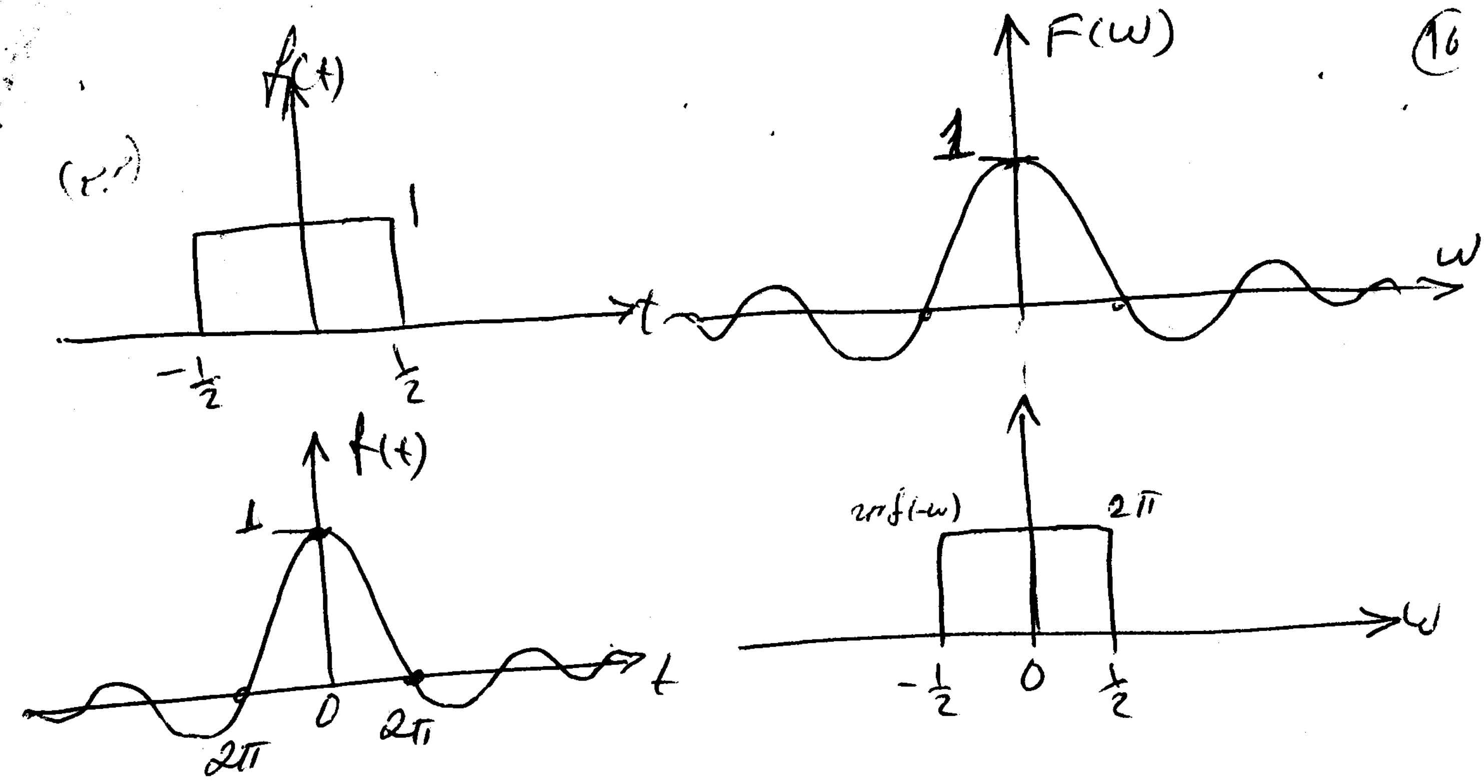
then $\mathcal{F}\{F(t)\} = 2\pi f(-\omega)$. ——— (4.1)

Example: If it's given that $\mathcal{F}\{rect(t)\} = Sa(\omega/2)$, determine $\mathcal{F}\{Sa(t/2)\}$.

Solution: Let $F(\omega) = Sa(\omega/2)$: then $F(t) = Sa(t/2)$

Using Eq (4.1) we can write

$$\mathcal{F}\{F(t)\} = 2\pi\, rect(-\omega) = 2\pi\, rect(\omega)$$



Duality of the Fourier transformation.

5) Coordinate Scaling

The expansion or compression of a time waveform effects the specoral density of the waveform.

The function $f(at)$ represent the function $f(t)$ compressed in the time scale by a facoor of (a). Similarly, a function $F(\frac{\omega}{a})$ represents a function $F(\omega)$ expanded in the frequency scale by the same factor (a).

$$\mathcal{F}\{f(at)\} = \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

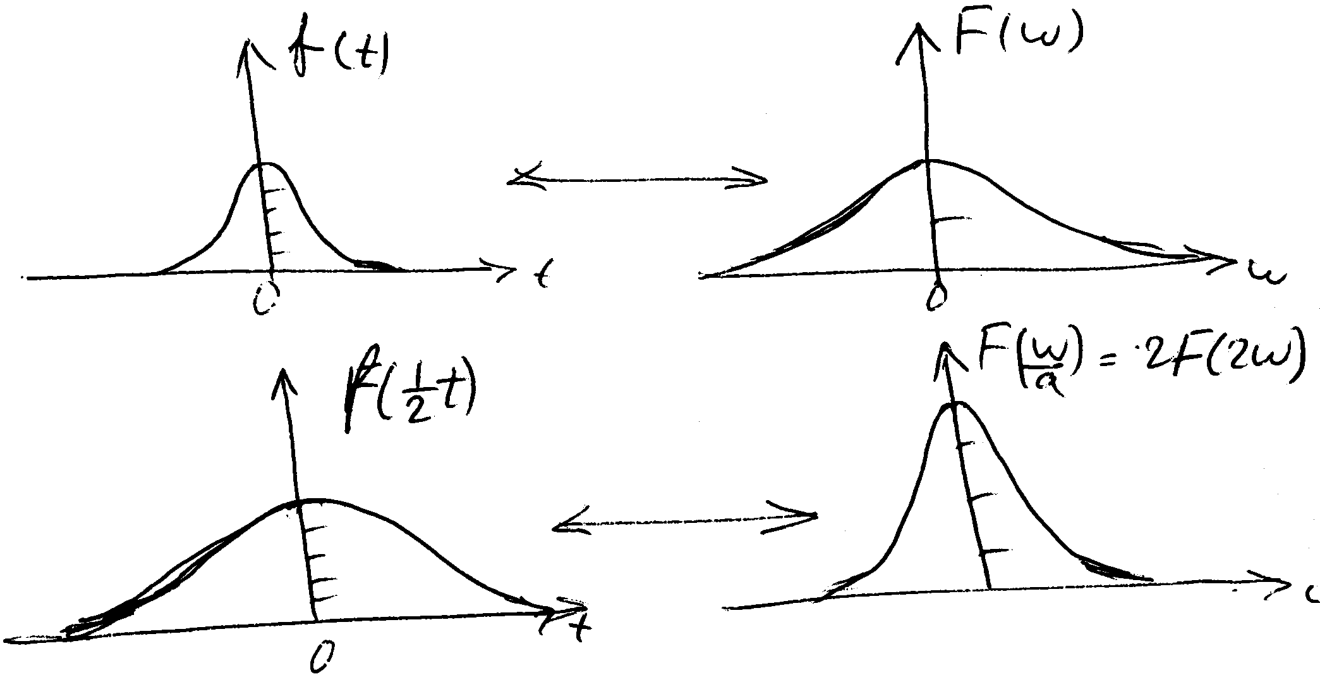
Proof:

$$\mathcal{F}\{f(at)\} = \int_{-\infty}^{\infty} f(at)\, e^{-j\omega t}\, dt$$

After changing the variable of integration to $x = at$, we have

$$\mathcal{F}\{f(at)\} = \int_{-\infty}^{\infty} f(x)\, e^{-j\frac{\omega x}{a}}\, dx/a$$

$$= \frac{1}{a} F\left(\frac{\omega}{a}\right) \quad \text{for } a > 0$$



Effects of time scale changes on the signal spectral density.

6) Time shifting (Delay)

This operation is a translation of the time origin, causing the signal to be delayed (or advanced) in time by some time t_0. The corresponding effect on the signal spectral density is:

$$\mathcal{F}\{f(t-t_0)\} = F(\omega)e^{-j\omega t_0}$$

Proof:

$$\mathcal{F}\{f(t-t_0)\} = \int_{-\infty}^{\infty} f(t-t_0)\, e^{-j\omega t}\, dt$$

Changing the variable of integration, let $x = t - t_0$

$$\mathcal{F}\{f(t-t_0)\} = \int_{-\infty}^{\infty} f(x)\, e^{-j\omega(x+t_0)}\, dx =$$

$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} f(x)\, e^{-j\omega x}\, dx =$$

$$= e^{-j\omega t_0} F(\omega).$$

Thus if a signal $f(t)$ is delayed in time by t_0 its magnitude spectral density remains unchanged and a nagative phase $(-\omega t_0)$ is add to each frequency component.

7). Frequency Shifting (Modulation).

The dual of the delay property is the frequency-translation property.

$$\mathcal{F}\{f(t)\, e^{j\omega_0 t}\} = F(\omega - \omega_0)$$

Proof: $$\mathcal{F}\{f(t)\, e^{j\omega_0 t}\} = \int_{-\infty}^{\infty} f(t)\, e^{j\omega_0 t} \cdot e^{-j\omega t}\, dt$$

(18) $$= \int_{-\infty}^{\infty} f(t)\, e^{-j(\omega-\omega_0)t}\, dt = F(\omega-\omega_0)$$

Therefore multiplying a time function by $e^{j\omega_0 t}$ causes its spectral density to be translated in frequency by ω_0 rad/sec.

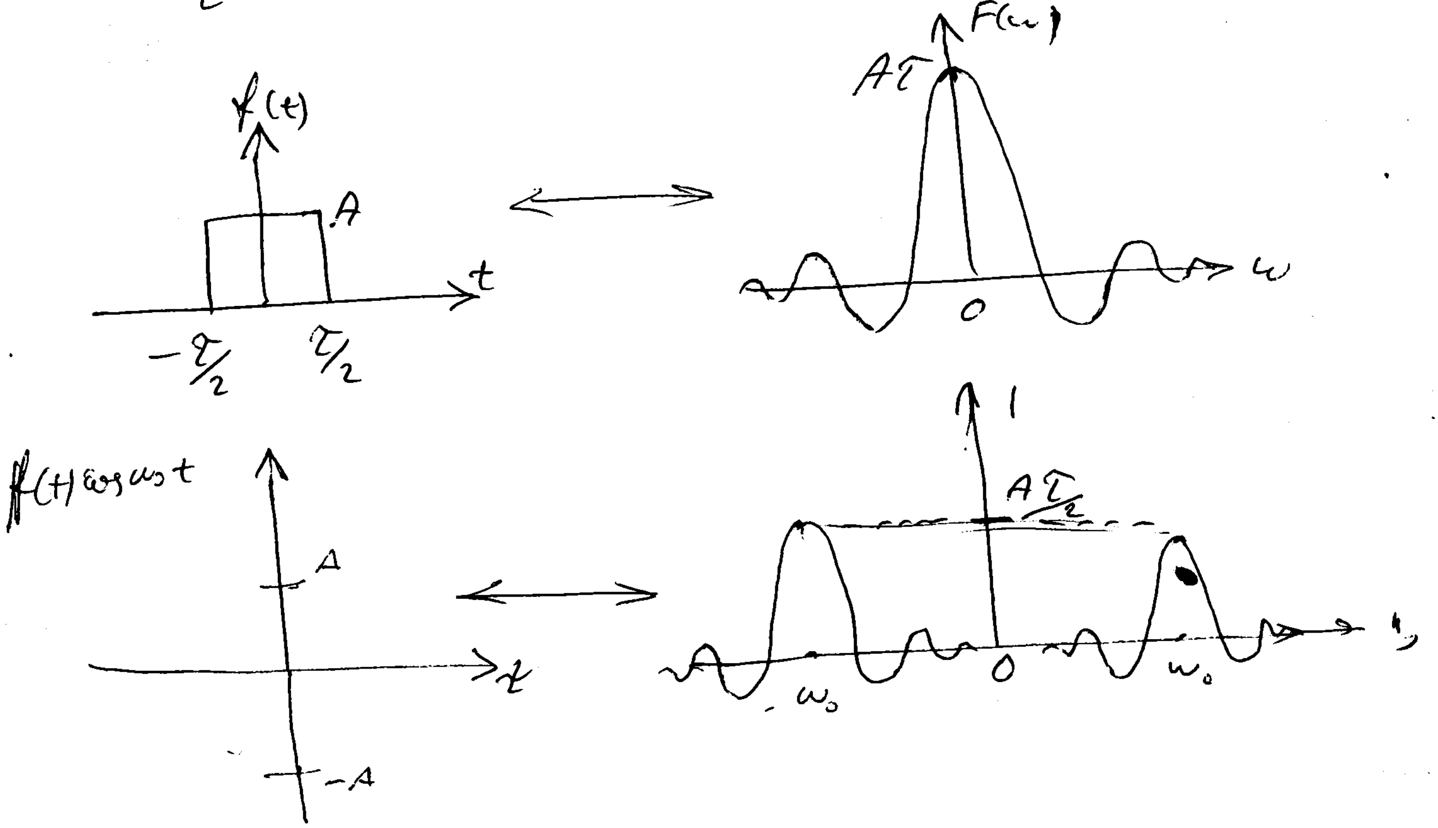
For real-valued $f(t)$. It is now a relatively simple matter to find the Fourier transform of $Re\{f(t)e^{j\omega_0 t}\}$?

Ex. Find the spectral density of the pulse waveform $A\,rect(t/\tau)\cos\omega_0 t$.

Solution $\mathcal{F}\{A\,rect(t/\tau)\} = A\tau\,Sa(\omega\tau/2)$.

Use of the modulation property gives

$$\mathcal{F}\{A\,rect(t/\tau)\cos\omega_0\tau\} = \frac{1}{2}A\tau\{Sa[(\omega+\omega_0)\tau/2] + Sa[(\omega-\omega_0)\tau/2]\}$$



Effects of modulation on the signal spectral density.

(19)

8) Differentiation and Integration

If $\frac{d}{dt} f(t)$ is absolutely integrable, then

$$\frac{d}{dt} f(t) \longleftrightarrow j\omega F(\omega)$$

Proof:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t} dt$$

$$F(\omega) = -\frac{1}{j\omega} f(t)\, e^{-j\omega t} \Big|_{-\infty}^{\infty} + \frac{1}{j\omega} \int_{-\infty}^{\infty} \frac{df(t)}{dt} e^{-j\omega t} dt =$$

0

$f(t)$ is Fourier transformable $\lim_{t \to \pm\infty} g(t) = 0$

so the first term is zero. and we get the desired result.

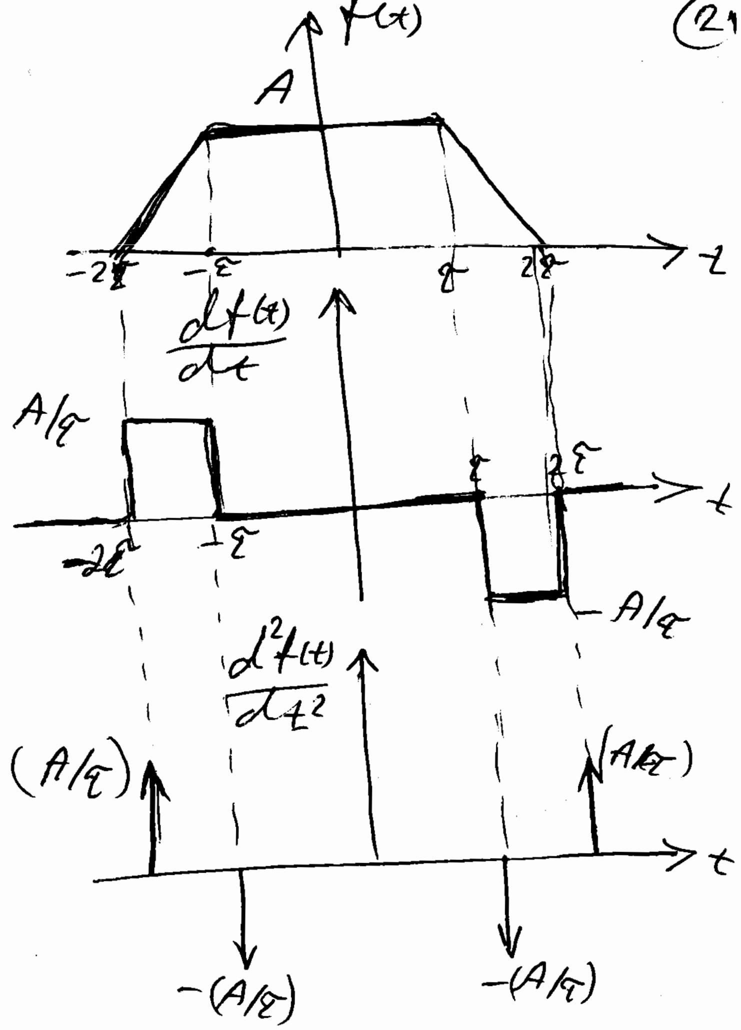
$$F(\omega) = \frac{1}{j\omega} \int_{-\infty}^{\infty} \frac{df}{dt} e^{-j\omega t} dt =$$

$$j\omega F(\omega) \Leftrightarrow \frac{d f(t)}{dt}.$$

In general form

$$\frac{d^n f}{dt^n} \leftrightarrow (j\omega)^n F(\omega).$$

Example: Determine the Fourier transform of the trapezoidal pulse shown in Fig bellow.



$$\frac{d^2 f(t)}{dt^2} = \frac{A}{\tau}\left[\delta(t+2\tau) - \delta(t+\tau) - \delta(t-\tau) + \delta(t-2\tau)\right];$$

$$(j\omega)^2 F(\omega) = \frac{A}{\tau}\left(e^{j2\omega\tau} - e^{j\omega\tau} - e^{-j\omega\tau} + e^{-j2\omega\tau}\right)$$

$$= \frac{A}{\tau}\left(e^{j\omega\tau/2} - e^{-j\omega\tau/2}\right)^2\left(e^{j\omega\tau} + 1 + e^{-j\omega\tau}\right)$$

$$F(\omega) = A\tau Sa^2(\omega\tau/2)[1+2\cos\omega\tau)$$

9) Convolution

There are two methods ~~of~~ characterizing a system, one is by it's frequency transfer function, a second method is by it's impulse response. We now wish to relate these methods using the principle of convolution.

For the test signal $f(t) = \delta(t-\tau)$, the system impulse response is defined as:

$$f\{\delta(t-\tau)\} = h(t,\tau)$$

where τ is the delay or „age" variable. and if the system is time-invariant, $h(t,\tau)$ takes the special form

$$f\{\delta(t-\tau)\} = h(t-\tau).$$

The input signal $f(t)$ may be expressed in terms of impulse function by.

$$f(t) = \int_{-\infty}^{\infty} f(\tau)\,\delta(\tau-t)\,d\tau = \int_{-\infty}^{\infty} f(\tau)\,\delta(t-\tau)\,d\tau$$

The impulse responses of the linear System corresponding to each value of the (τ) is

$$g(t) = f\left\{\int_{-\infty}^{\infty} f(\tau).\,\delta(t-\tau)\,d\tau\right\}.$$

from the integration theory, we can rewrite this as:-

$$g(t) = f\left\{ \lim_{\Delta\tau \to 0} \sum_{n=-\infty}^{\infty} f(\tau_n)\,\delta(t-\tau_n)\,\Delta\tau \right\}.$$

Using the principle of superposition. we move the system operator inside the summation.

$$g(t) = \lim_{\Delta\tau \to 0} \sum_{n=-\infty}^{\infty} f(\tau_n) \cdot f\{\delta(t-\tau_n)\}\,\Delta\tau.$$

But $f\{\delta(t-\tau)\} = h(t,\tau) = h(t-\tau)$

so $g(t) = \int_{-\infty}^{\infty} f(\tau)\cdot h(t-\tau)\,d\tau$ — after returning to the integral form.

This is a key result in signal analysis for it links the input to the output by means of an integral operation and holds for any linear system.

This result is known as the <u>convolution integral</u>.

$$\int_{-\infty}^{\infty} f(\tau)\,h(t-\tau)\,d\tau \triangleq f(t) \circledast h(t).$$

If $\mathcal{F}\{f(t)\} = F(\omega)$, $\mathcal{F}\{h(t)\} = H(\omega)$

Then $\mathcal{F}\{f(t) \circledast h(t)\} = F(\omega)\cdot H(\omega).$

Proof:-

$$\mathcal{F}\{f(t)\circledast h(t)\} = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\tau)\,h(t-\tau)\,d\tau\right]e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty} f(\tau)\underbrace{\left[\int_{-\infty}^{\infty} h(t-\tau)\,e^{-j\omega t}\,dt\right]}_{\mathcal{F}\{h(t-\tau)\}}d\tau$$

Using the time-shift property we have:

$$\mathcal{F}\{h(t-\tau)\} = e^{-j\omega\tau}\cdot H(\omega).$$

Then $$\mathcal{F}\{f(t)\circledast h(t)\} = \int_{-\infty}^{\infty} f(\tau).H(\omega)e^{-j\omega\tau}\,d\tau$$

$$= H(\omega)\cdot\int_{-\infty}^{\infty} f(\tau)\,e^{-j\omega\tau}\,d\tau =$$

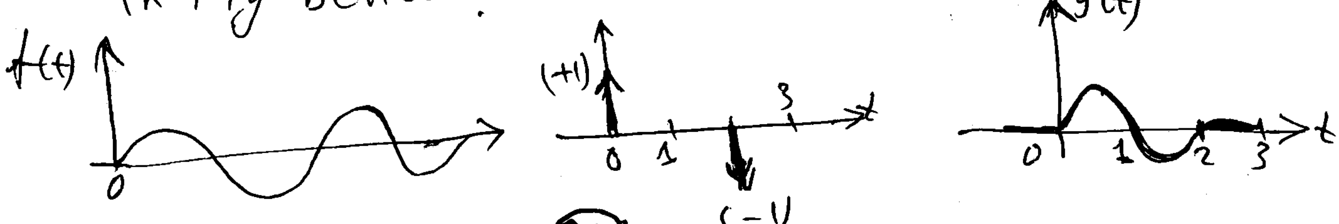
$$= H(\omega)\cdot F(\omega).$$

V Thus convolution in the time domain corresponds to multiplication in the frequency domain.

Using the dual property we have

$$F_1(\omega)\circledast F_2(\omega) = \int_{-\infty}^{\infty} F_1(u)\cdot F_2(\omega-u)\,du.$$

Example: Find $f(t)\circledast h(t)$ for the $f(t)$, $h(t)$ shown in Fig bellow.



$f(t) = A\sin \pi t\, u(t)$, $\quad h(t) = \delta(t) - \delta(t-2)$.

$$g(t) = f(t) \circledast h(t) = \int_{-\infty}^{\infty} [A\sin \pi\tau\, u(\tau)][\delta(t-\tau) - \delta(t-2-\tau)]\, d\tau$$

$$= [A\sin \pi t]\, u(t) - [A\sin \pi(t-2)]\, u(t-2)$$

$$g(t) = \begin{cases} 0 & t < 0 \\ A\sin \pi t & 0 < t < 2 \\ 0 & t > 2 \end{cases}$$