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قسم
هندسة اتصالات
المرحلة الثانية

Communication I lectuers

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الاتصالات

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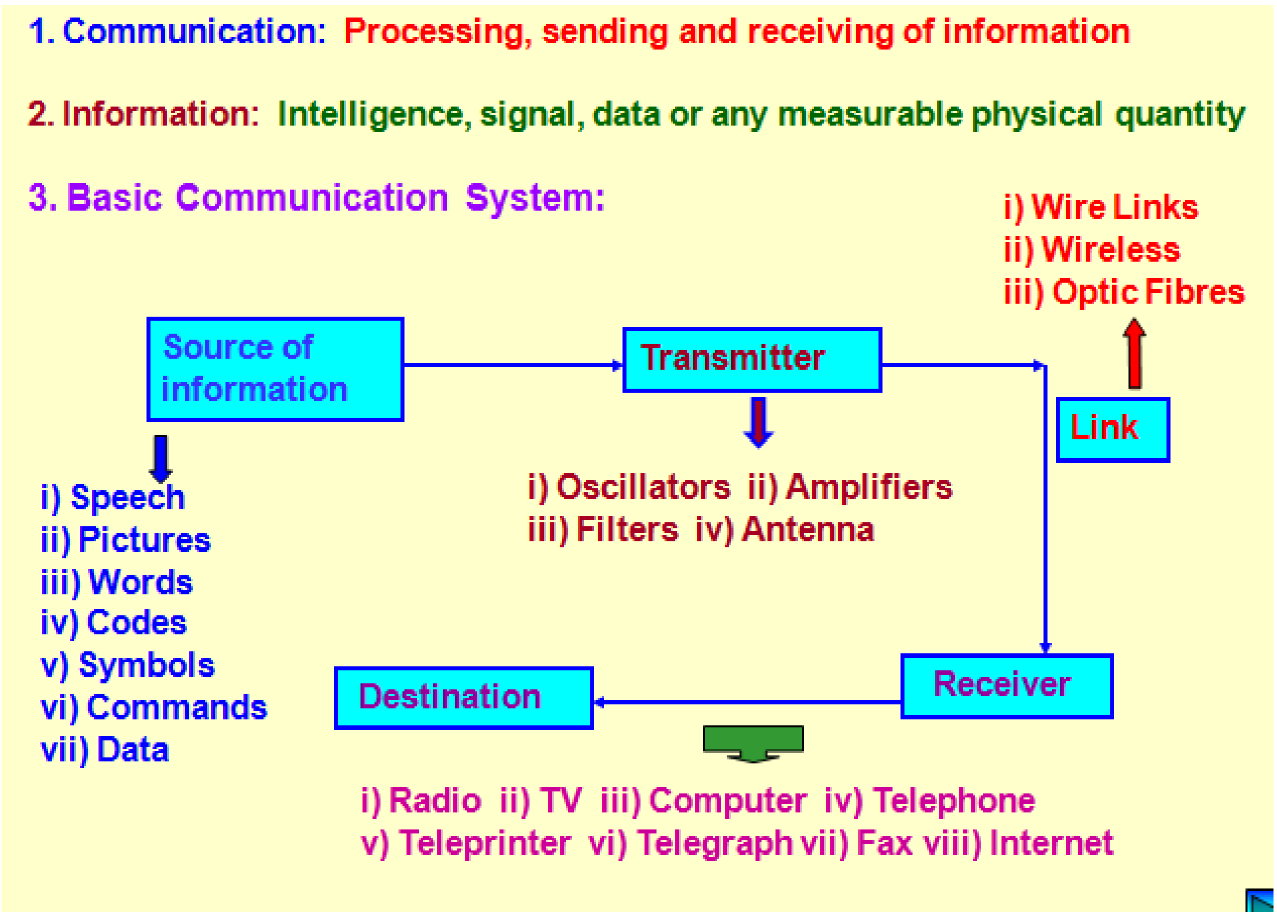
Chapter One

Signal and its Fourier Series Representations

Chapter one

Signals and Systems in communication

1.1 introduction;



Forms of Communication:

1. Radio Broadcast
2. Television Broadcast
3. Telephony
4. Telegraphy
5. Radar
6. Sonar
7. Fax (Facsimile Telegraphy)
8. E-mail
9. Teleprinting
10. Telemetering
11. Mobile Phones
12. Internet

Types of communication:

1. Cable communication
2. Ground wave communication
3. Sky wave communication
4. Satellite communication
5. Optic fibre communication

1.2 The Signals

Any time varying physical phenomenon that can convey information is called signal. Some examples of signals are human voice, electrocardiogram, sign language, videos etc. There are several classification of signals such as Continuous time signal, discrete time signal and digital signal, random signals and non-random signals.

1.2 classifications of signals

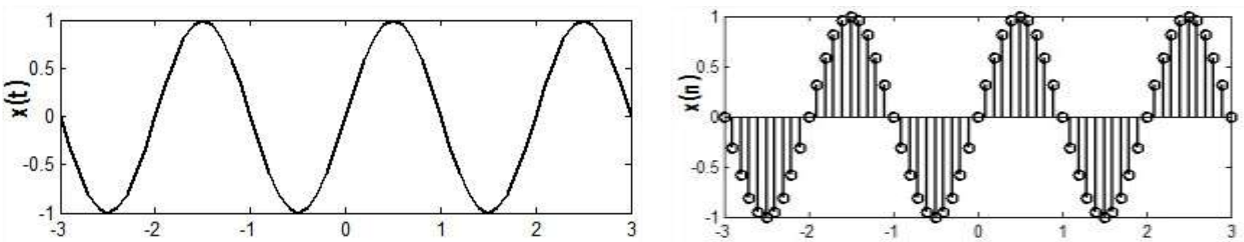
a. Analog signals

(1) Continuous-time Signal:

A continuous-time signal is a signal that can be defined at every instant of time. A continuous-time signal contains values for all real numbers along the X-axis. It is denoted by *x(t)*. Figure 1(a) shows continuous-time signal.

(2) Discrete-time Signal:

Signals that can be defined at discrete instant of time is called discrete time signal. Basically discrete time signals can be obtained by sampling a continuous-time signal. It is denoted as *x(n)*.Figure 1(b) shows discrete-time signal.



a b

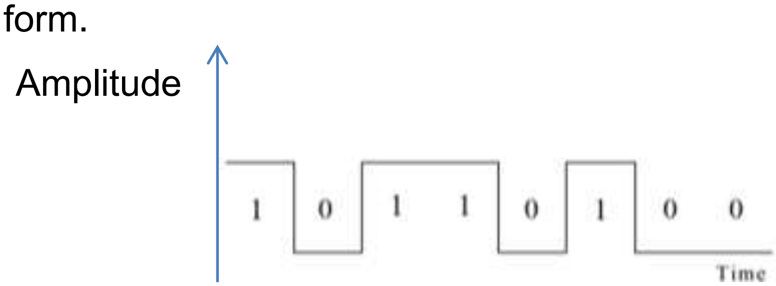
Fig.1(a) Continuous-time signal (b) Discrete-time signal

The signals of Continuous vs. Discrete may be the simplest classification to understand as the idea of discrete-time and continuous-time is one of the most fundamental properties to all of signals and system.

A system where the input and output signals are continuous is a continuous system , and one where the input and ouput signals are discrete is a discrete system .

b. Digital Signal:

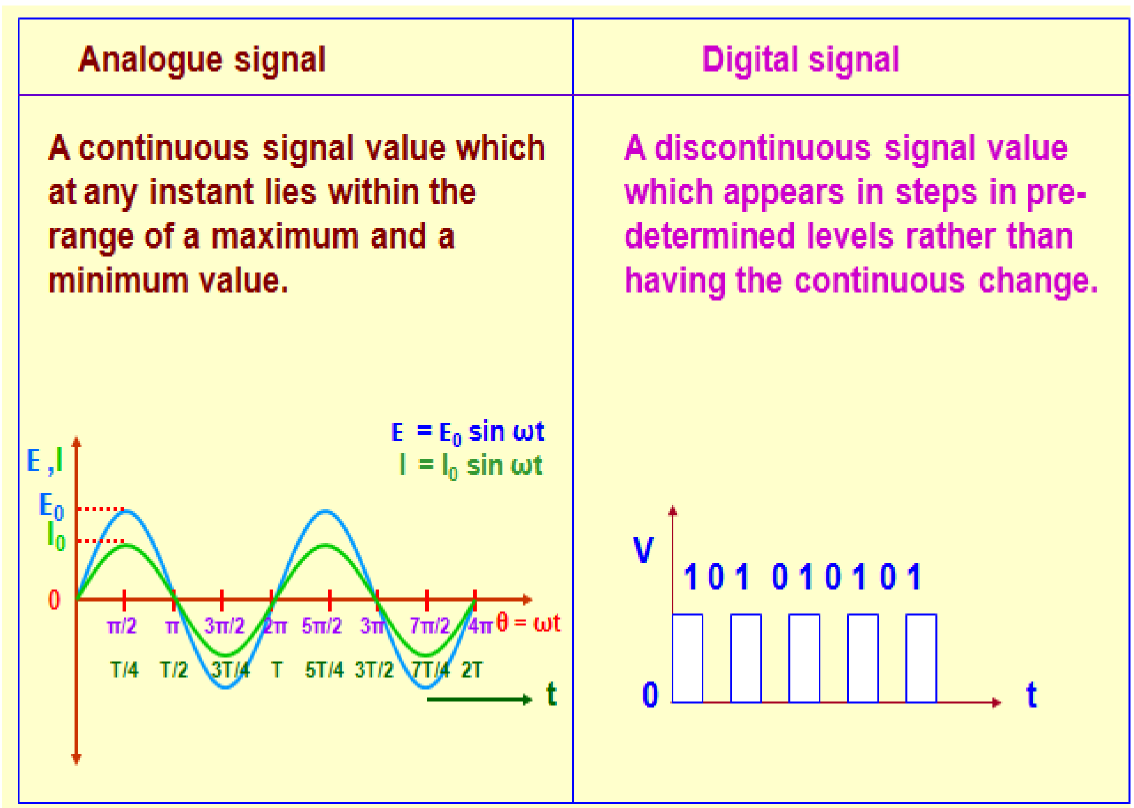
The signals that are discrete in time and quantized in amplitude are called digital signal. The term "digital signal" applies to the transmission of a sequence of values of a discrete-time signal in the form of some digits in the encoded form.



Fig(2) binary digital signal

Analog vs. Digital

The difference between analog and digital is similar to the difference between continuous time and discrete-time. In this case, however, the difference is with respect to the value of the function (y-axis). Analog corresponds to a continuous y-axis, while digital corresponds to a discrete y-axis. An easy example of a digital signal is a binary sequence, where the values of the function can only be one or zero.



c. Periodic and Aperiodic Signal:

A signal is said to be periodic if it repeats itself after some amount of time $x(t+T)=x(t)$, for some value of T. The period of the signal is the minimum value of time for which it exactly repeats itself.

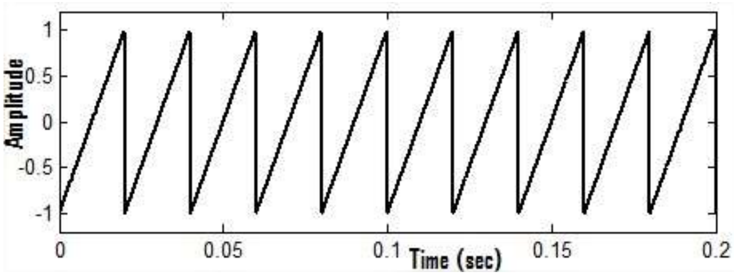


Fig.3(a) Periodic signal

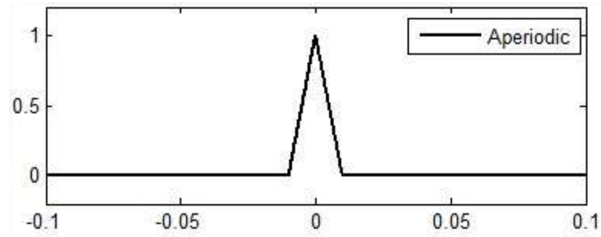
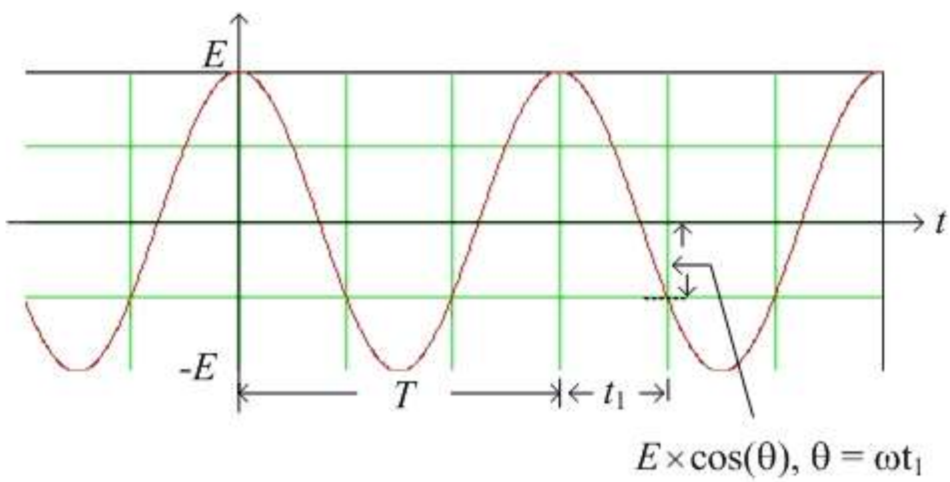


Fig.3(b) Aperiodic signal

Signal which does not repeat itself after a certain period of time is called aperiodic signal. The periodic and aperiodic signals are shown in Figure 3(a) and 3(b) respectively.

The topic of sinusoidal signal has already been introduced. Hence the description is kept brief. The waveform of a sinusoidal signal is shown in Fig. 4.

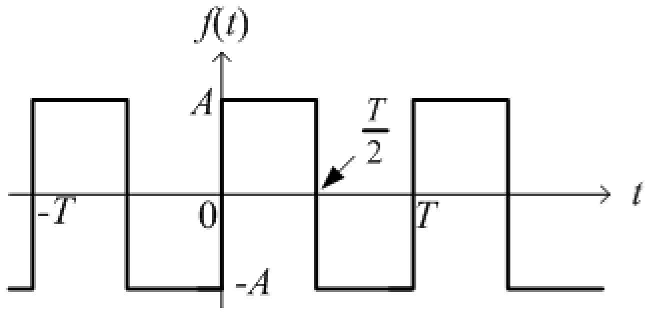


Fig(4) periodic sinusoidal signal

A periodical signal, $f(t)$ with a period of T, repeats itself every T seconds. A signal is periodic if

$$f(t) = f(t \pm T)$$

A square wave-signal, shown in Fig. 5, is a periodic signal.



Fig(5) aperiodic symmetric square-wave signal

Periodic signals repeat with some period T, while aperiodic, or nonperiodic, signals do not.

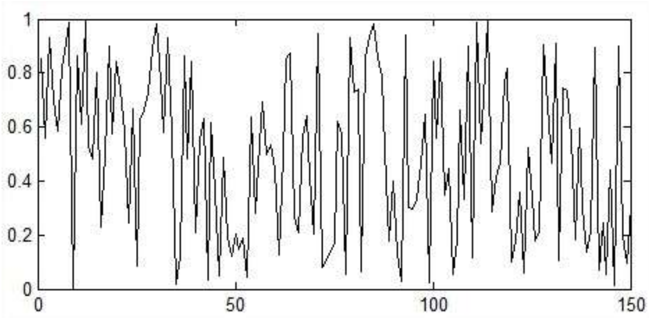
We can define a periodic function through the following mathematical expression, where t can be any number and T is a positive constant:

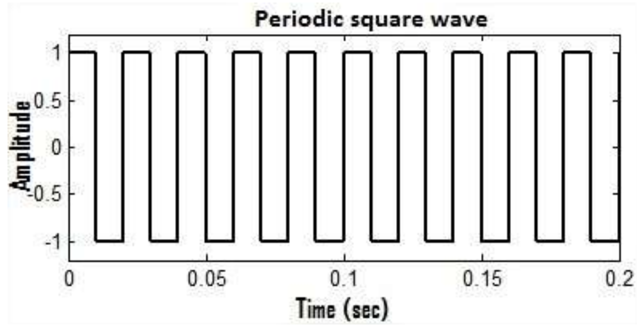
$f(t) = f(T + t)$

The fundamental period of our function, f (t), is the smallest value of T that the still allows the above equation, Equation 2.7, to be true.

d. Random and Deterministic Signal:

A random signal cannot be described by any mathematical function, where as a deterministic signal is one that can be described mathematically. A common example of random signal is noise. Random signal and deterministic signal are shown in the Figure 6(a) and 6(b) respectively.





a b

Fig.6(a) Random signal Fig.6(b) Deterministic signal

A signal whose physical description is known completely, in either a mathematical form or a graphical form is said to be a deterministic signal.
A signal that is known only in terms of probabilistic description, such as mean value, mean squared value, and so on, is said to be a random signal.
Most of the noise signals encountered in practical situations are random signals.
All message signals in communication systems are random signals, because some uncertainty (randomness) about the message must exists such that the signal conveys information from the sender to the receiver.

e. Causal, Non-causal and Anti-causal Signal:

Signal that are zero for all negative time, that type of signals are called causal signals, while the signals that are zero for all positive value of time are called anti-causal signal.
A non-causal signal is one that has non zero values in both positive and negative time. Causal, non-causal and anti-causal signals are shown below in the Figure 7(a), 7(b) and 7(c) respectively.

A causal system is one that is nonanticipative ; that is, the output may depend on current and past inputs, but not future inputs. All ”realtime” systems must be causal, since they can not have future inputs available to them.

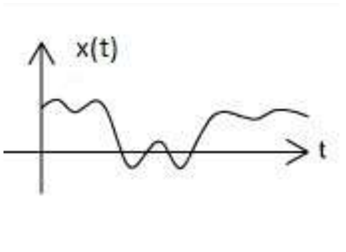


Fig.7(a) Causal signal

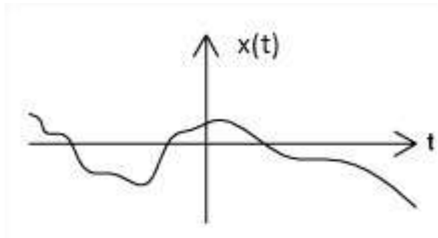


Fig.7(b)Non-causal signal

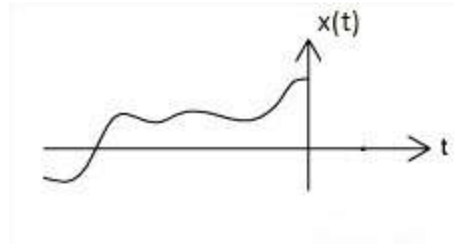


Fig.7(c) Anti-causal signal

Causal signals are signals that are zero for all negative time, while anitcausal are signals that are zero for all positive time. Noncausal signals are signals that have nonzero values in both positive and negative time

f. Even and Odd Signal:

An even signal is any signal '*x*' such that *x(t) = x(-t)*. On the other hand, an odd signal is a signal '*x'* for which *x(t) = -x(-t)*. Even signals are symmetric around the vertical axis, so that they can easily spotted.

Figure 8(a) and 8(b) shows the odd signal and even signal respectively.

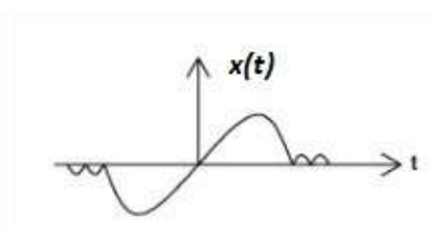


Fig.8(a) Odd signal

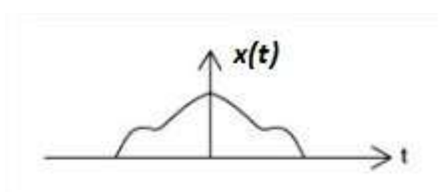


Fig.8(b) Even signal

A signal x(t) is said to be an even signal its value is symmetrical about the vertical axis, i.e. x(-t) = x(t)

A signal x(t) is said to be an odd signal if its value is anti-symmetrical about the horizontal axis, i.e. x(-t) = - x(t)

Examples of even and odd signals are shown in the figure (a) and (b), respectively.

1.3 energy and power signals

1.3.1 Energy signal

Some signals qualify to be classified as energy signals, whereas some other signals qualify to be classified as power signals. Given a continuous-time signal *f*(*t*), the energy contained over a finite time interval is defined as follows.

$$E_{(T_1,T_2)} = \int_{T_1}^{T_2} \left|f(t)\right|^2 \cdot dt, \;\; T_2 > T_1$$

$$E_f = \int_{-\infty}^{\infty} \left|f(t)\right|^2 \cdot dt$$

The first Equation defines the energy contained in the signal over time interval from T_1 till T_2. On the other hand, the second equation defines the *total energy* contained in the signal. If the total energy of a signal is a finite non-zero value, then that signal is classified as an *energy signal*. Typically the signals which are not periodic turns out to be energy signals. For example, a single rectangular pulse and a decaying exponential signal are energy signals.

1.3.2 Power signal

When a reference to power in a signal is made, it points to the *average power*. Power is defined as energy per second. For a continuous-time signal, we can obtain an expression for power from equation above .

$$P_f = \underset{T\to\infty}{\text{Limit}} \frac{1}{2T} \int_{-T}^{T} |f(t)|^2 \cdot dt,$$

$$P_f = \frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2 \cdot dt, \; T = \text{cycle period}$$

Most of the periodic signals tend to be power signals. Given the period of a cycle, the power of a periodic signal can be defined by first equation . and can be used to find the power of a dc signal also. The dc signal is also a power signal. If power of a signal is a finite non-zero value and its energy is infinite, then that signal is classified as a power signal. There are some signals which can be classified neither as power signals nor as energy signals. For example, a ramp signal defined from zero till infinity is neither a power signal nor an energy signal, since both power and energy of ramp signal is not bounded. But in practice. such a signal cannot exist and hence such a signal is not of any practical importance.

Example 1:

Given an exponential signal as defined by equation (3.12), find its energy.

$$f(t) = A \times \exp(-k \cdot t) \cdot \mathrm{u}(t)$$

Solution:

$$E_f = \int_{-\infty}^{\infty} |f(t)|^2 \cdot dt = \int_{0}^{\infty} \left(A \cdot e^{-kt}\right)^2 \cdot dt = \frac{A^2}{2k}$$

An exponential signal is an energy signal, since its energy is a finite, non-zero value

Example 2

Given a sinusoidal signal as defined by equation (3.14), find its power.

$$f(t) = A \times \sin(\omega \cdot t) \cdot$$

Solution:

$$E_f = \frac{1}{T} \int_{0}^{T} |f(t)|^2 \cdot dt = \frac{1}{T} \int_{0}^{T} \left(A \cdot \sin(\omega \cdot t)\right)^2 \cdot dt = \frac{A^2}{2}$$

The solution is expressed by equation (3.15). A sinusoidal signal is a power signal, since the its power over a cycle is a finite, non-zero value. The energy associated with the sinusoidal signal is infinite.

Example 3

Given a square-wave signal as defined by equation (3.15), find its power.

$$f(t) = -A \quad \text{for } -T/2 < t < 0$$
$$= A \quad \text{for } 0 < t < T/2$$

Solution:

$$E_f = \frac{1}{T}\int_{-T/2}^{T/2} |f(t)|^2 \cdot dt = \frac{1}{T} \times \int_{-T/2}^{T/2} A^2 \cdot dt = A^2$$

The solution is expressed by equation (3.18). A periodic square-wave signal is a power signal, since the its power over a cycle is a finite, non-zero value. The energy associated with the square-wave signal is infinite.

Example 4

determine if the following signals are Energy signals, Power signals, or neither, and evaluate *E* and *P* for each signal

a) $a(t) = 3\sin(2\pi t),\ -\infty < t < \infty$,

This is a periodic signal, so it must be a power signal. Let us prove it.

$$E_a = \int_{-\infty}^{\infty} |a(t)|^2\ dt = \int_{-\infty}^{\infty} |3\sin(2\pi t)|^2\ dt$$
$$= 9\int_{-\infty}^{\infty} \frac{1}{2}[1-\cos(4\pi t)]dt$$
$$= 9\int_{-\infty}^{\infty} \frac{1}{2}dt - 9\int_{-\infty}^{\infty} \cos(4\pi t)dt$$
$$= \infty \quad \mathrm{J}$$

Notice that the evaluation of the last line in the above equation is infinite because of the first term. The second term has a value between –2 to 2 so it has no effect in the overall value of the energy.

Since $a(t)$ is periodic with period T = $2\pi/2\pi$ = 1 second, we get

$$P_a = \frac{1}{1}\int_0^1 |a(t)|^2\, dt = \int_0^1 |3\sin(2\pi t)|^2\, dt$$

$$= 9\int_0^1 \frac{1}{2}\left[1-\cos(4\pi t)\right]dt$$

$$= 9\int_0^0 \frac{1}{2}dt - 9\int_0^1 \cos(4\pi t)dt$$

$$= \frac{9}{2} - \left[\frac{9}{4\pi}\sin(4\pi t)\right]_0^1$$

$$= \frac{9}{2} \quad \text{W}$$

So, the energy of that signal is infinite and its average power is finite (9/2). This means that it is a power signal as expected. Notice that the average power of this signal is as expected (square of the amplitude divided by 2)

b) $b(t) = 5e^{-2|t|},\ -\infty < t < \infty$,

Let us first find the total energy of the signal.

$$E_b = \int_{-\infty}^{\infty} |b(t)|^2\, dt = \int_{-\infty}^{\infty} \left|5e^{-2|t|}\right|^2\, dt$$

$$= 25\int_{-\infty}^{0} e^{4t}dt + 25\int_0^{\infty} e^{-4t}dt$$

$$= \frac{25}{4}\left[e^{4t}\right]_{-\infty}^{0} + \frac{25}{4}\left[e^{-4t}\right]_0^{\infty}$$

$$= \frac{25}{4} + \frac{25}{4} = \frac{50}{4} \quad \text{J}$$

The average power of the signal is

$$P_b = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} |b(t)|^2\ dt = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} \left|5e^{-2|t|}\right|^2\ dt$$
$$= 25\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{0} e^{4t}dt + 25\lim_{T\to\infty}\frac{1}{T}\int_{0}^{T/2} e^{-4t}dt$$
$$= \frac{25}{4}\lim_{T\to\infty}\frac{1}{T}\left[e^{4t}\right]_{-T/2}^{0} + \frac{25}{4}\lim_{T\to\infty}\frac{1}{T}\left[e^{-4t}\right]_{0}^{T/2}$$
$$= \frac{25}{4}\lim_{T\to\infty}\frac{1}{T}\left[1-e^{-2T}\right] + \frac{25}{4}\lim_{T\to\infty}\frac{1}{T}\left[e^{-2T}-1\right]$$
$$= 0+0=0$$

So, the signal $b(t)$ is definitely an energy signal.
So, the energy of that signal is infinite and its average power is finite (9/2). This means that it is a power signal as expected. Notice that the average power of this signal is as expected (the square of the amplitude divided by 2)

c) $$c(t) = \begin{cases} 4e^{+3t}, & |t| \le 5 \\ 0, & |t| > 5 \end{cases},$$

d) $$d(t) = \begin{cases} \dfrac{1}{\sqrt{t}}, & t > 1 \\ 0, & t \le 1 \end{cases},$$

Let us first find the total energy of the signal.

$$E_d = \int_{-\infty}^{\infty} |d(t)|^2\ dt = \int_{1}^{\infty}\frac{1}{t}\ dt$$
$$= \ln\left[t\right]_1^{\infty}$$
$$= \infty - 0 = \infty \quad \text{J}$$

So, this signal is NOT an energy signal. However, it is also NOT a power signal since its average power as shown below is zero.

The average power of the signal is

$$P_d = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |d(t)|^2 \, dt = \lim_{T\to\infty} \frac{1}{T} \int_{1}^{T/2} \frac{1}{t} dt$$

$$= \lim_{T\to\infty} \left(\frac{1}{T} \ln[t]_1^{T/2} \right) = \lim_{T\to\infty} \left(\frac{1}{T} \ln\left[\frac{T}{2}\right] - \frac{1}{T} \ln[1] \right)$$

$$= \lim_{T\to\infty} \left(\frac{1}{T} \ln\left[\frac{T}{2}\right] \right) = \lim_{T\to\infty} \left(\frac{\ln\left[\frac{T}{2}\right]}{T} \right)$$

Using Le'hopital's rule, we see that the power of the signal is zero. That is

$$P_d = \lim_{T\to\infty} \left(\frac{\ln\left[\frac{T}{2}\right]}{T} \right) = \lim_{T\to\infty} \left(\frac{\frac{2}{T}}{1} \right) = 0$$

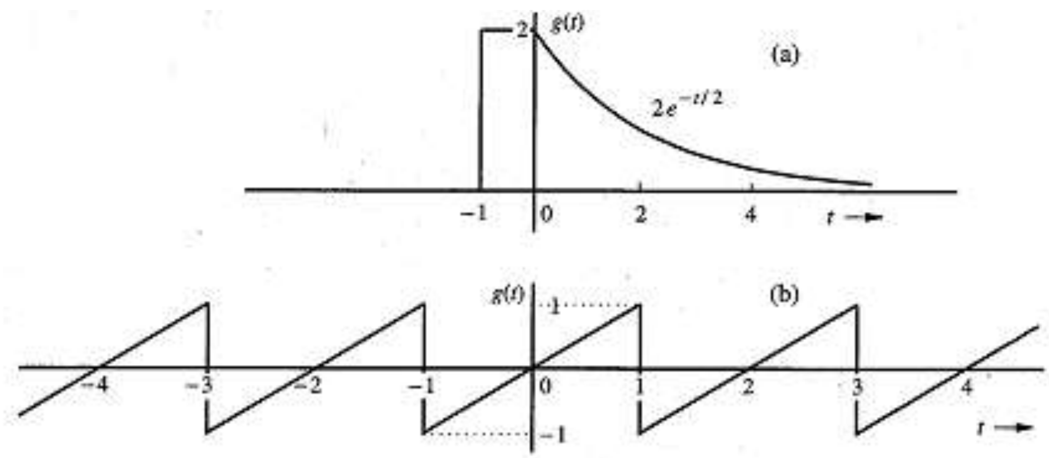
So, not all signals that approach zero as time approaches positive and negative infinite is an energy signal. They may not be power signals either.

e) $e(t) = -7t^2, \quad -\infty < t < \infty$,

f) $f(t) = 2\cos^2(2\pi t), \quad -\infty < t < \infty$.

g) $g(t) = \begin{cases} 12\cos^2(2\pi t), & -8 < t < 31 \\ 0, & \text{elsewhere} \end{cases}$.

Example: Which of the following signals is an energy signal and which is a power signal?



Properties of Energy and Power Signals

A signal with finite energy has a zero power, i.e. an energy signal cannot be a power signal.

A signal with finite power has an infinite energy, i.e. a power signal cannot be an energy signal.

Therefore, a signal cannot be both an energy signal and a power signal.

On the other hand, some signals are neither energy signals nor power signals; the ramp signal is such an example.

Every signal that is generated in the lab is an energy signal, i.e. every signal in real life is an energy signal.

On the other hand, a power signal must necessarily have an infinite duration and an infinite energy such that its average power reaches a finite (non-zero) value in the limit; a true power signal is therefore impossible to generate in real life.

All the periodic signals for which the energy in one period is finite are power signals.

However, not all power signals are periodic; there exists some power signals that are aperiodic.

Example:

Classify the following signal(s) as a power signal, energy signal, or neither and find its power or energy
as appropriate x(t) = e−t;

Solution:

$$x(t) = e^{-t}$$

$$E_x = \int_{-\infty}^{\infty} \left|x^2(t)\right| dt = \int_{-\infty}^{\infty} e^{(-2t)} dt = \frac{-e^{-2t}}{2}\Big|_{-\infty}^{\infty} = \infty$$

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 \, dt = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{(-2t)} dt = \lim_{T\to\infty} \frac{1}{T} \left[\frac{-e^{(-2t)}}{2}\right]_{-\frac{T}{2}}^{\frac{T}{2}} = \infty$$

∴ Since this signal has infinite energy and power signals, it is neither an energy nor a power signal.

1.4 Time-Shifting of Signal:

In signals and system amplitude scaling, time shifting and time scaling are some important properties. If a continuous time signal is defined as *x(t) = s(t - t1)*. Then we can say that *x(t)* is the time shifted version of *s(t)*.

Consider a simple signal *s(t)* for *0 < t < 1*

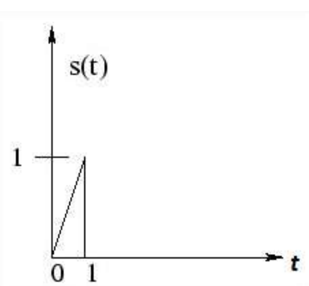


Fig.8(a) Signal within *0< t < 1*

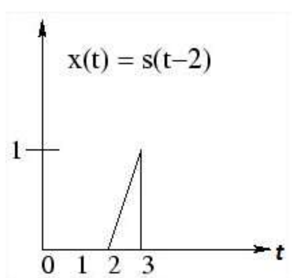


Fig.8(b) Signal shifted by *2* sec.

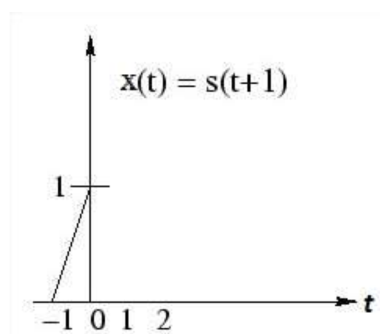


Fig.8(c) Signal shifted by -1 sec.

Now shifting the function by time $t1 = 2$ sec.

$x(t) = s(t-2) = t-2 \quad$ for $\quad 0 < (t-2) < 1$

$= t-2 \quad$ for $\quad 2 < (t-2) < 3$

Which is simply signal $s(t)$ with its origin delayed by 2 sec.

Now if we shift the signal by $t1 = -1$ sec.

then $x(t) = s(t+1) = t+1 \quad$ for $\quad 0 < (t+1)$

$= t+1 \quad$ for $\quad -1 < t < 0$.

Which is simply $s(t)$ with its origin shifted to the left or advance in time by 1 seconds. This time-shifting property of signal is shown in the Figure 8(a), 8(b) and 8(c) given above.

1.5 Time-Scaling of Signal:

Time scaling compresses or dilates a signal by multiplying the time variable by some quantity. If that quantity is greater than one, the signal becomes narrower and the operation is called compression. If that quantity is less than one, the signal becomes wider and the operation is called dilation. Figure 7(a), 7(b), 7(c) shows the signal x(t), compression of signal and dilation of signal respectively.

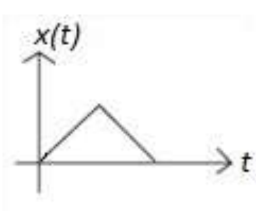


Fig.7(a) Signal x(t)

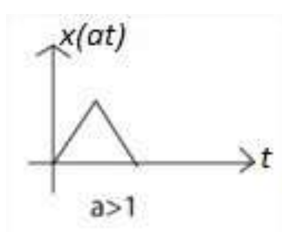


Fig.7(b) Compression of signal

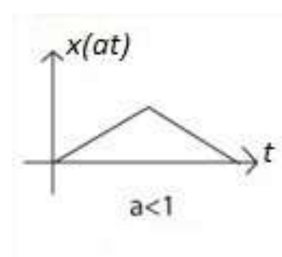
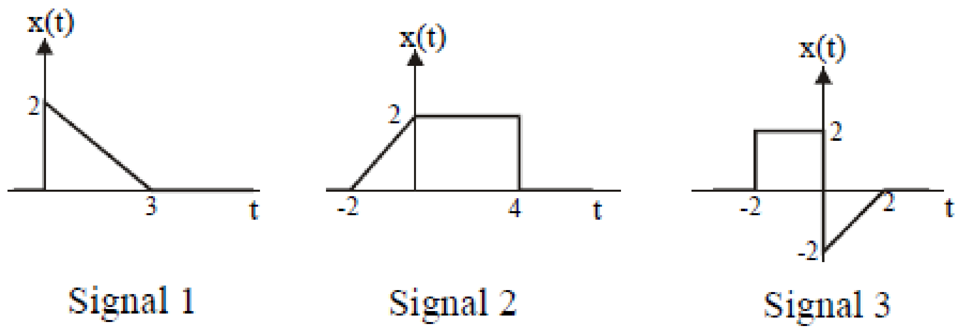


Fig.7(c) Dilation of signal

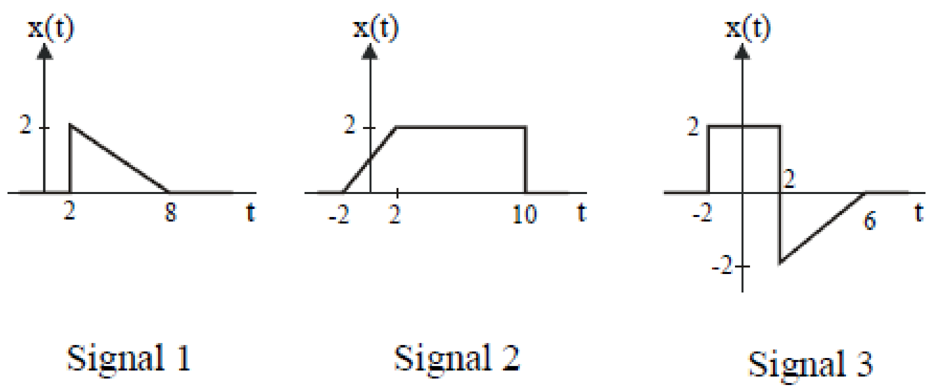
For each signal x(t) of Figure below ,



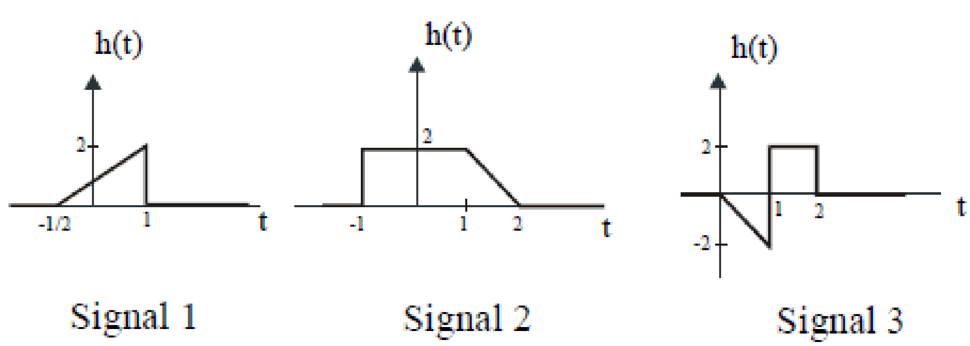
sketch the following: (a) p(t) = x [0.5 (t − 2)]; (b) h(t) = x (2 − 2t); (c) s(t) = x (−0.5t − 1)

Solution:

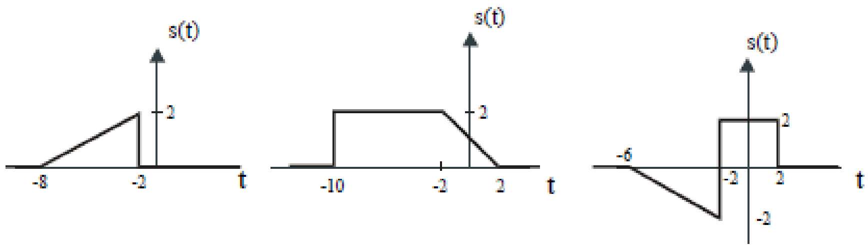
(a) p(t) = x [0.5 (t − 2)]



(b) h(t) = x (2 − 2t) = x (−2t+2) = x [−2 (t − 1)]



(c) $s(t) = x(-0.5t - 1) = x[-0.5(t+2)] = x[-0.5(t - (-2))]$



Signal 1 Signal 2 Signal 3

Time inversion of signal

The time inversion of a signal $g(t)$ is denoted by $g(-t)$ and is given by

$$\phi(t) = g(-t) \qquad (1\text{-}13)$$

This is the mirror image of $g(t)$ about the vertical axic as shown in (Fig 1.17).
Therefor to time invert a signal we replace (t) with $-t$. Thus, the time inversion of signal $g(t)$ yields $g(-t)$

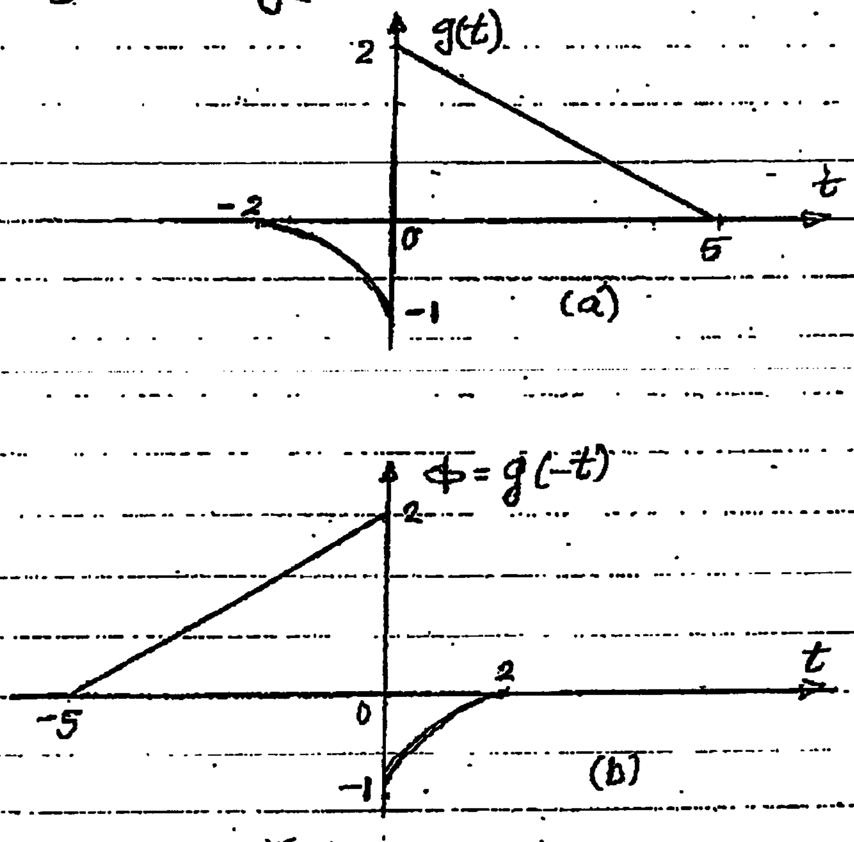


Fig. 1.17

Ex: 1.4

For the signal in Fig. 1.18a which is $g(t)$, sketch $g(-t)$

Solution see (Fig 1.18b)

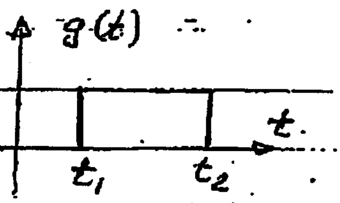


Fig 1.18a

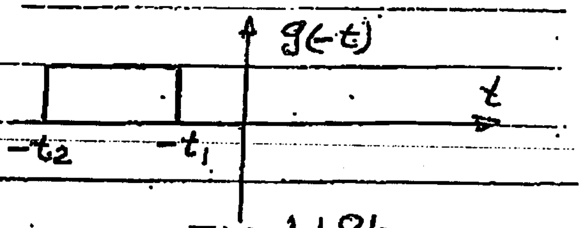


Fig. 1.18b

Ex. 1.5 For the signal $g(t)$ shown in Fig. 1.19a, sketch $g(t)$

Solution: see Fig. 1.19.b

If $g(t) = e^{t/2}$ then

$g(-t) = e^{-t/2}$

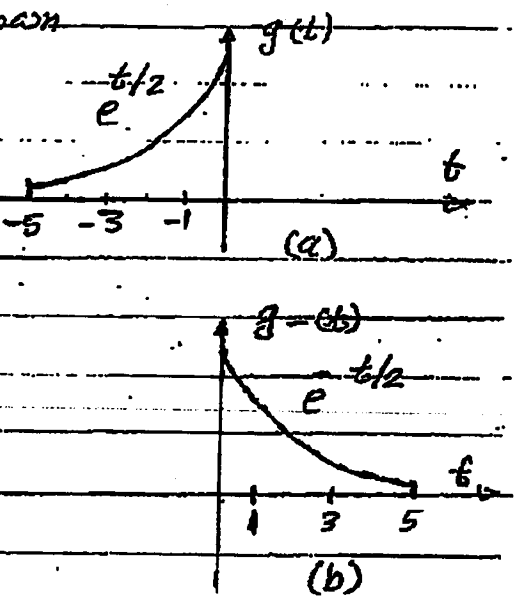


Fig. 1.19

~~If there are more than one operation included then~~ ~~the operations must be performed from left to right.~~

Ex. 1.6

(i) $g(t-c)$

(ii) $g(k(t-c))$

(iii) $g(-k(t-c))$

in (i) there is one operation which is "shifting" by c

in (ii) there are two operation : time-scaling by k and time-shifting by c. We perform first the time-scaling followed by time shifting.

in ex. (iii)

$$g(\underset{\text{inverse}}{-}\underset{\text{scaling}}{k}\underbrace{(t-c)}_{\text{shifting}})$$

there are three operations from left to right

first time inverse

Second time-scaling by k

and at last time-shifting b c

Ex. 1.7 For the signal shown in Fig. 1-20 sketch $g(6-2t)$

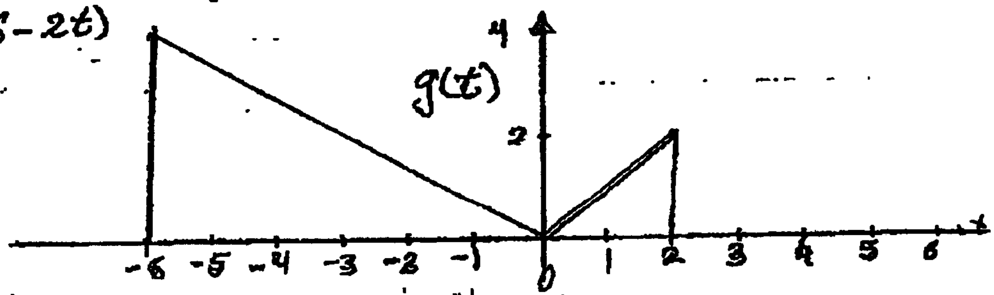


Fig 1-20

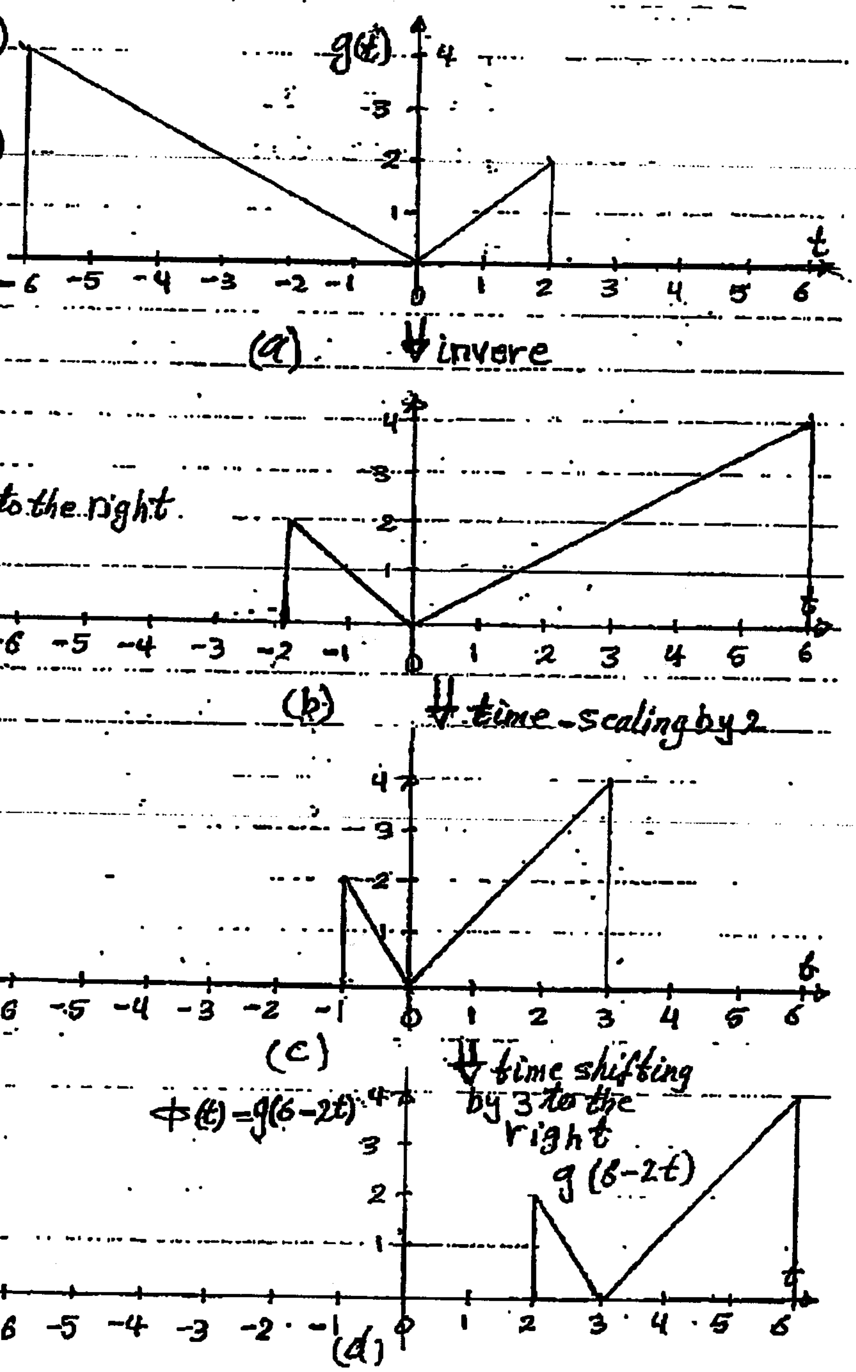
Solution of Ex 1.7

For the time-shifting we have to modifly the opertion to a form like t - c. Hence

$g(6-2t) = g(-(2t-6))$

$= g(-2(t-3))$

$\therefore$ the signal form is



$g(-2(t-3))$

$-$: time invers; 2 : time scaling; $t-3$: time shifting to the right.

See Fig. 1.21

Fig. 1.21

Note that $\Phi(3) = g(0)$

$\Phi(2) = g(2)$

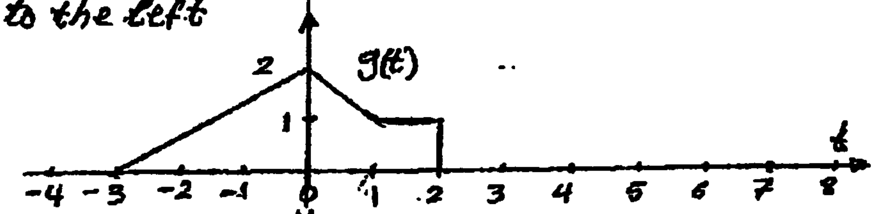
$\Phi(6) = g(-6)$

Ex. 1.8: For the signal shown in Fig 1-22 sketch

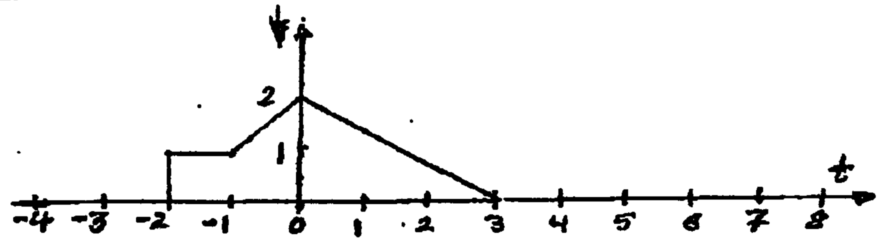
$$\Phi(t) = g\left(-1-\frac{t}{2}\right)$$

Solution:

$$\Phi(t) = g\Big(\underbrace{-}_{\text{time inverse}}\underbrace{\tfrac{1}{2}}_{\text{scaling by 1/2}}\underbrace{(t+2)}_{\text{shifting by 1 to the left}}\Big)$$

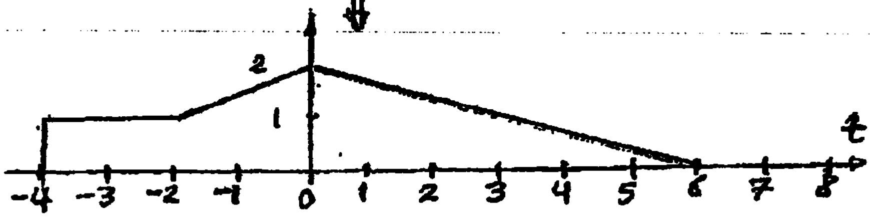


time invers



Scaling by 1/2 (time extention)

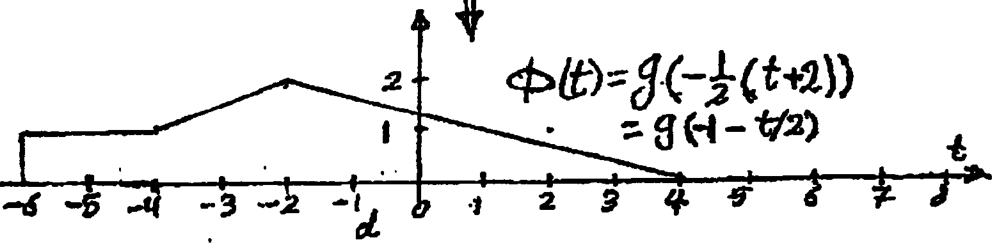
Fig. 1-22



shifting to the left by 2

$\Phi(-6) = g(2)$
$\Phi(-4) = g(1)$
$\Phi(-2) = g(0)$
$\Phi(4) = g(-3)$

$$\Phi(t) = g\left(-\tfrac{1}{2}(t+2)\right) = g(-1-t/2)$$



1.3 Singularity function (Unit impulse function or Dirac delta $\delta(t)$, Unit step function $U(t)$)

Singularity functions are mathematical idealization and, strictly speaking, they do not occure in physical systems. Members of such functions are Unit Impulse function (also called impulse function or Dirac delta) $\delta(t)$ and Unit step function $U(t)$. They have simple mathematical forms. Because of these and other properties, it is possible often to evaluate complicated expression which might other wise be impossible (or at least very difficult) to solve.

Unit Impulse function $\delta(t)$

This function is one of the most important functions in the study of signals and systems. We can visualize an impulse as an rectangular pulse whose width approach zero (0), its hight approach infinity (∞) but its area always equal to Unity (1). We sall draw an arraw as an Indicator of the impulse function as shown in (Fig. 1.23)

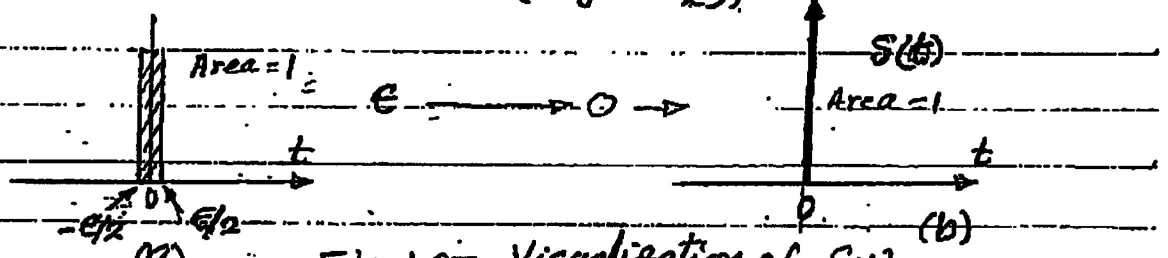


Fig. 1.23 Visualization of $\delta(t)$

If $\delta(t)$ is shifted by (t_0) to the right or to the left then its loction will be as shown in Fig. 1.24

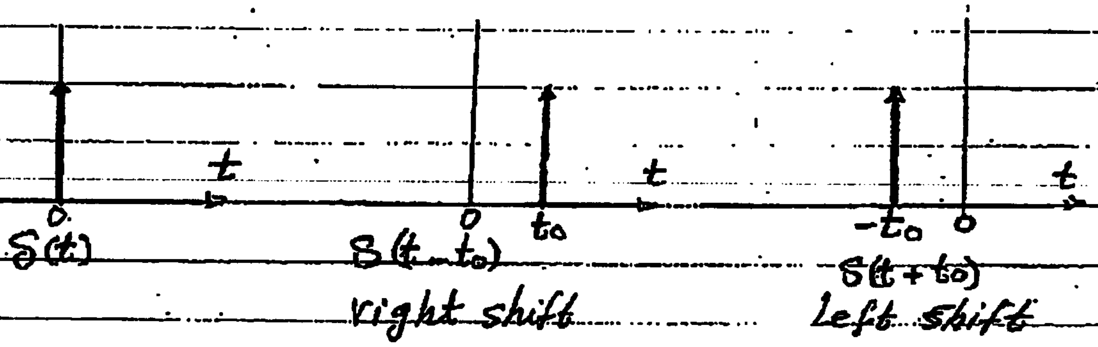


Fig 1.24 location of the impulse

Properties of the impulse function

1- $\delta(t) = 0$ for $t \neq 0$ (1.14a)

2- $\delta(t-t_0) = 0$ for $t \neq t_0$ (1.14b)

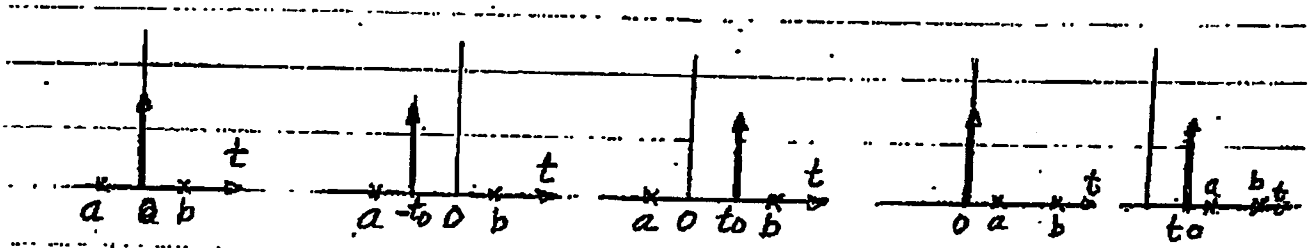
3- $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$ $-\infty < 0 < \infty$ (1-15a)

4- $\int_{a}^{b} \delta(t)\, dt = 1$ $a < 0 < b$ (1-15b)

5- $\int_{-\infty}^{\infty} \delta(t-t_0)\, dt = 1$ $-\infty < t_0 < \infty$ (1-16a)

6- $\int_a^b \delta(t-t_0)\, dt = 1 \qquad a < t_0 < b \qquad (1\text{-}16b)$

Eqs (1.15b and 1.16b) indicat that the value of integration (the area of the impulse) becomes unity when the impulse located inside the interagion limits and zero (0) when it locates ouside the integation limits as shown in (Fig. 1.25).



(a) $\int_a^b \delta(t)dt = 1$ — Location inside integral limits

$\int_a^b \delta(t+t_0)dt = 1$ — Location inside Integral limits

$\int_a^b \delta(t-t_0)dt = 1$ — Location inside integral limits

$\int_a^b \delta(t)dt = 0$ — Location outside Integral limits

$\int_a^b \delta(t+t_0)dt = 0$ — outside Integral limits

Fig. 1.25

Symmetry: $\delta(t)$ function is an even function $\quad \delta(t) = \delta(-t) \quad (1\text{-}17a)$

Time scaling: $\delta(at) = \frac{1}{|a|}\delta(t) \quad (1\text{-}17b)$

Multiplication of a Function by an impulse

1- $f(t)\,\delta(t) = f(0)\,\delta(t) \qquad$ 1-18a

2- $f(t)\,\delta(t-t_0) = f(t_0)\,\delta(t-t_0) \qquad$ 1-18b

Sampling Property of unit impulse Function

1- $$\int_{-\infty}^{\infty} f(t)\,\delta(t)\,dt = \int_{-\infty}^{\infty} f(0)\,\delta(t)\,dt$$

$$= f(0)\int_{-\infty}^{\infty} \delta(t)\,dt = f(0) \qquad (1.19a)$$

2- $$\int_{-\infty}^{\infty} f(t)\,\delta(t-t_0)\,dt = \int_{-\infty}^{\infty} f(t_0)\,\delta(t-t_0)\,dt$$

$$= f(t_0)\int_{-\infty}^{\infty} \delta(t-t_0) = f(t_0) \qquad (1.19b)$$

3- $$\int_{a}^{b} f(t)\,\delta(t-t_0)\,dt = f(t_0) \text{ if } a<t_0<b \qquad (1.19c)$$

Unit Step Function

Another familiar and useful function is the "Unit step function $u(t)$" defined by see (Fig. 1.25)

$$u(t) = \begin{cases} 1 & t>0 \\ 0 & t<0 \end{cases} \qquad (1.20a)$$

and

$$u(t-t_0) = \begin{cases} 1 & t>t_0 \\ 0 & t<t_0 \end{cases} \qquad (1.20b)$$

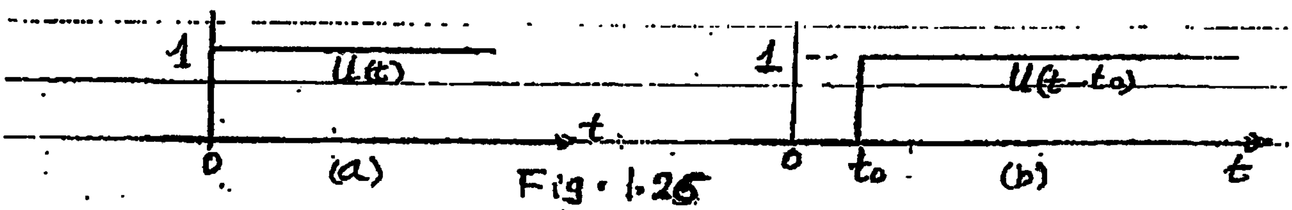


Fig. 1.25

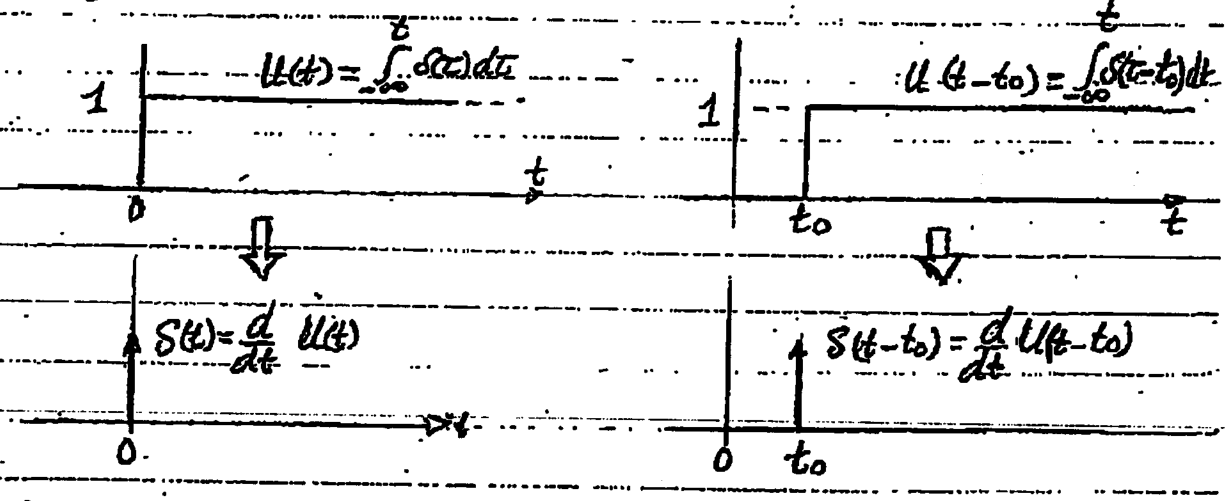
Relationship of Impulse function to the Unit step function:

1. $$\int_{-\infty}^{t} \delta(\tau)\, d\tau = u(t) \qquad (1\text{-}31)$$

2. $$\int_{-\infty}^{t} \delta(\tau - t_0)\, d\tau = u(t - t_0) \qquad (1\text{-}32)$$

3. $$\frac{d}{dt}\, u(t) = \delta(t) \qquad (1\text{-}33)$$

4. $$\frac{d}{dt}\, u(t - t_0) = \delta(t - t_0) \qquad (1\text{-}34)$$



(Fig 1.27) Differtiation of Unit step function

~~Ex 1.9 simplify the following~~ expression

(a) $\dfrac{\sin t}{t^2+2}\,\delta(t)$

(b) $\dfrac{1}{j\omega+2}\,\delta(\omega+3)$

Solution

see eqs. (1.18, p 32)

(a) $\dfrac{\sin t}{t^2+2}\,\delta(t) = \left.\dfrac{\sin t}{t^2+2}\right|_{t=0} \delta(t) = \dfrac{\sin 0}{0+2}\,\delta(t) = 0$

(b) $\dfrac{1}{j\omega+2}\,\delta(\omega+3) = \left.\dfrac{1}{j\omega+2}\right|_{\omega=-3} \cdot \delta(\omega+3) = \dfrac{1}{-3j+2}\,\delta(\omega+3)$

Ex. 1.10

Evaluate the following integrals

(a) $\int_{-\infty}^{\infty} \delta(t-3)\, e^{-t}\, dt$

(b) $\int_{-\infty}^{\infty} \log t\; \delta(t-10)\, dt$

(c) $\int_{-\infty}^{\infty} \delta(t-2)\cos \pi(t-3)\, dt$

(d) $\int_{1}^{10} e^{-x^2}\, \delta(x)\, dx$

(e) $\int_{0}^{100} \cos x\pi\; \delta(x+2)\, dx$

(f) $\int_{-10}^{10} \cos x\pi\; \delta(x+2)\, dx$

(g) $\int_{-\infty}^{\infty} (t+6)\,\delta(-3t+9)\,dt$ (h) $\int_{-4}^{\infty} t^2 \cos t\pi\,\delta(2t+4)\,dt$

Solution

see eqs. (1.15a,b / 1-16a,b / 1-17a,b) and Fig. 1.25 P. 31,32

(a) $\int_{-\infty}^{\infty} \delta(t-3)\,e^{-t}\,dt$ hier $a=-\infty$, $b=\infty$, $t=3$
Hence $a<t<b$

$\therefore \int_{-\infty}^{\infty} \delta(t-3)\,e^{-t}\,dt = e^{-t}\Big|_{t=3} = e^{-3} = 0.0497$

(b) $\int_{-\infty}^{\infty} \log t\,\delta(t-10)\,dt$ hier $a=-\infty$, $b=\infty$ $t=10$
therefor $a<t<b$

$\therefore \int_{-\infty}^{\infty} \log t\,\delta(t-10)\,dt = \log t\Big|_{t=10} = \log 10 = 1$

(c) $\int_{-\infty}^{\infty} \delta(t-2)\cos\pi(t-3)\,dt$ hier $a=-\infty$, $b=\infty$, $t=3$
therefor $a<t<b$

$\therefore \int_{-\infty}^{\infty} \delta(t-2)\cos\pi(t-3)\,dt = \cos\pi(t-3)\Big|_{t=3} = \cos 0 = 1$

(d) $\int_{1}^{10} e^{-x^2}\,\delta(x)\,dx$ hier $a=1$ $b=10$ $x=0$
therfor x lies outside the interval between a and b

$\therefore \int_{1}^{10} e^{-x^2}\,\delta(x)\,dx = 0$

(e) $\int_{0}^{100} \cos x\pi \, \delta(x+2)\,dx$ here $a=0$, $b=100$, $x=-2$

since $x=-2$ outside the interval betwee a and b, therefore

$$\int_{0}^{100} \cos x\pi \, \delta(x+2)\,dx = 0$$

(f) $\int_{-10}^{10} \cos x\pi \, \delta(x+2)\,dx$ $a=-10$, $b=10$, $x=-2$

x lies inside the interval $[a,b]$

$$\therefore \int_{-10}^{10} \cos x\pi \, \delta(x+2)\,dx = \cos x\pi \Big|_{x=-2} = \cos(-2\pi) = \cos 2\pi = 1$$

(g) $\int_{-\infty}^{\infty} (t+6)\,\delta(-3t+9)\,dt$ $a=-\infty$, $b=\infty$, $t=3$

since t lies inside the interval $[a,b]$

$$\therefore \int_{-\infty}^{\infty} (t+6)\,\delta(-3t+9)\,dt = \int_{-\infty}^{\infty} (t+6)\,\delta(-(3t-9))\,dt$$

since δ is an even functio, then $\delta(-t) = \delta(t)$

$$\therefore \delta(-(3t-9)) = \delta(3t-9)$$

$$\therefore \int_{-\infty}^{\infty} \text{--} = \int_{-\infty}^{\infty} (t+6)\,\delta(3t-9)\,dt = \int_{-\infty}^{\infty} (t+6)\,\delta(3(t-3))\,dt$$

since $\delta(at) = \frac{1}{|a|}\delta(t)$, the $\delta(3(t-3)) = \frac{1}{3}\delta(t-3)$

$$\therefore \int_{-\infty}^{\infty} \text{---} = \int_{-\infty}^{\infty} (t+6) \times \frac{1}{3}\,\delta(t-3)\,dt = \frac{1}{3}\int_{-\infty}^{\infty} (t+6)\,\delta(t-3)\,dt$$

$$\frac{1}{3}\int_{-\infty}^{\infty}(t+6)\,\delta(t-3)\,dt = \frac{1}{3}(t+6)\Big|_{t=3} = \frac{9}{3} = 3$$

$$\therefore \int_{-\infty}^{\infty}(t+6)\,\delta(-3t+9)\,dt = 3$$

(h) $\int_{-4}^{\infty} t^2 e^{\sin t\pi/4} \cos t\pi \, \delta(2t-4)$

here $a=-4$, $b=\infty$, $t=2$, since t lies inside the intrval $[a,b] = (-4,\infty)$, we evaluate the integral.

$$\int_{-4}^{\infty} t^2 e^{\sin t\pi/4} \cos t\pi \, \delta(2t-4)\,dt = \int_{-4}^{\infty} t^2 e^{\sin t\pi/4} \cos t\pi \, \delta(2(t-2))\,dt$$

$$= \int_{-4}^{\infty} t^2 e^{\sin t\pi/4} \cos t\pi \times \frac{1}{2}\delta(t-2)\,dt$$

$$= \frac{1}{2} t^2 e^{\sin t\pi/4} \cos t\pi \Big|_{t=2}$$

$$= \frac{1}{2} \times 2^2 e^{\sin \pi/2} \cos 2\pi$$

$$= 2\,e^{-1} \times 1 = \frac{2}{e} = 0.7357$$

Ex. 1.11

Determine the power of the following signals

(a) $g(t) = C\cos(\omega_0 t + \theta)$

(b) $g(t) = C_1\cos n\omega_0 t + C_2\cos m\omega_0 t$ $\quad n, m$ integer

(c) $g(t) = D e^{j\omega_0 t}$

Solution

(a) $g(t) = C\cos(\omega_0 t + \theta)$

We shall find the power first by determining the average power over infinit long time using eq (1.5)

$$P_g = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |g(t)|^2 dt = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} C^2\cos^2(\omega_0 t + \theta)dt$$

$$= \lim_{T\to\infty}\frac{1}{T}\left[\int_{-T/2}^{T/2}\frac{C^2}{2}(1+\cos(2\omega_0 t + 2\theta)dt\right]$$

$$= \lim_{T\to\infty}\frac{C^2}{2T}\int_{-T/2}^{T/2} dt + \lim_{T\to\infty}\frac{C^2}{2T}\int_{-T/2}^{T/2}\cos(2\omega_0 t + 2\theta)dt$$

The firs term on the right-hand side is equal to $\frac{C^2}{2}$, that is

$$\lim_{T\to\infty}\frac{C^2}{2T}\int_{-T/2}^{T/2} dt = \lim_{T\to}\frac{C^2}{2T}(t)\Big|_{-T/2}^{T/2} = \frac{C^2}{2T}T = \frac{C^2}{2}$$

the second term is zero

$$\lim_{T\to\infty}\frac{C^2}{2T}\int_{-T/2}^{T/2}\cos(2\omega_0 t + 2\theta)dt = 0$$

~~because the integral represents the area under~~ a sinusoid over a very larg time interval T with $T \to \infty$. This area is at most equal to the area under half cycil because of cancelations of the positive and negative areas of a sinusoid

$$\therefore \boxed{P_g = C^2/2}$$

This shows that a sinusoid of amplitude C has a Power $C^2/2$. regardless of the value of the frequency ω_0 ($\omega_0 \neq 0$) and the phase θ.

Anothe approach for determining the averge power of periodic signal can be found using eq. (1-6) with the integral over one period time T_0 and $\omega_0 = 2\pi/T_0$

$$P_g = \frac{1}{T_0}\int_{-T_0/2}^{T/2} |g(t)|^2 \, dt = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} C^2 \cos^2(\omega_0 t + \theta) dt$$

$$= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \frac{C^2}{2}[1 + \cos(2\omega_0 t + 2\theta) dt$$

$$= \frac{C^2}{2T_0}\int_{-T_0/2}^{T_0/2} dt + \frac{C^2}{2T_0}\int_{-T_0/2}^{T_0/2} \cos(2\omega_0 t + 2\theta) dt$$

$$= \frac{C^2}{2T_0} t \Big|_{-T_0/2}^{T_0/2} + \frac{C^2}{2T_0} \frac{\sin(2\omega_0 t + 2\theta)}{2\omega_0}\Big|_{-T_0/2}^{T_0/2}$$

$$\boxed{P_g = \frac{C^2}{2}} \quad \text{since} \quad \sin(2\omega_0 t + 2\theta)\Big|_{-T_0/2}^{T_0/2} = 0$$

whic is similar to the result of first approach

~~(b) $g(t) = C_1 \cos n\omega_0 t + C_2 \cos m\omega_0 t$ n, m integer numbers~~

since $\cos n\omega_0 t$ and $\cos m\omega_0 t$ are periodic function and the frequency $(n\omega_0)$ and $(m\omega_0)$ are both a multiple of a common frequency ω_0 (Fundamental frequency), therfor the summation of them $g(t)$ is also a periodic function. Therefor we can use eq.(1-6) for caculating P_{av}

$\omega_0 = 2\pi/T_0$ where T_0 is the period of fundamental

$$P_{av} = \frac{1}{T_0} \int_0^{T_0} (C_1 \cos n\omega_0 t + C_2 \cos m\omega_0 t)^2 \, dt$$

$$= \frac{1}{T_0} \Big[\int_0^{T_0} C_1^2 \cos^2 n\omega_0 t \, dt + \int_0^{T_0} C_2^2 \cos^2 m\omega_0 t \, dt + \int_0^{T_0} 2 C_1 C_2 \cos n\omega_0 t \cos m\omega_0 t \, dt \Big]$$

The first integral

$$\frac{C_1^2}{T_0} \int_0^{T_0} \cos^2 n\omega_0 t \, dt = \frac{C_1^2}{T_0} \int_0^{T_0} \frac{1}{2}(1 + \cos 2n\omega_0 t) \, dt$$

$$= \frac{C_1^2}{2T_0} \Big[t \Big|_0^{T_0} + \frac{\sin 2n\omega_0 t}{2n\omega_0} \Big|_0^{T_0} \Big]$$

$$= \frac{C_1^2}{2T_0} \Big[T_0 - 0 + \Big(\frac{\sin 2n\omega_0 T_0}{2n\omega_0} - 0 \Big) \Big] = \frac{C_1^2}{2T_0} \Big[T_0 + \frac{\sin 2n \frac{2\pi}{T_0} T_0}{2n\omega_0} \Big]$$

$$= \frac{C_1^2}{2T_0} \Big(T_0 + \frac{\sin 4n\pi}{2n \, 2\pi/T_0} \Big) = \frac{C_1^2}{2} + \frac{C_1^2}{2} \Big(\frac{\sin 4n\pi}{4n\pi} \Big)$$

~~since $\sin 4n\pi = 0$ for $n = 1, 2, 3, \ldots$~~

$$\therefore \frac{1}{T_0}\int_0^{T_0} C_1^2 \cos^2 n\omega_0 t\, dt = \frac{C_1^2}{2} \quad \text{for } n \neq 0$$

in the same way the integral

$$\frac{1}{T_0}\int_0^{T_0} C_2^2 \cos^2 m\omega_0 t\, dt = \frac{C_2^2}{2} \quad \text{for } m \neq 0$$

The last integral

$$\frac{2C_1C_2}{T_0}\int_0^{T_0} \cos n\omega_0 t \cos m\omega_0 t\, dt$$

$$= \frac{2C_1C_2}{T_0}\left[\int_0^{T_0} \frac{1}{2}\left(\cos(n+m)\omega_0 t + \cos(n-m)\omega_0 t\right) dt\right]$$

$$= \frac{C_1C_2}{T_0}\left[\int_0^{T_0} \cos(n+m)\omega_0 t\, dt + \int_0^{T_0} \cos(n-m)\omega_0 t\, dt\right]$$

the first integral is always equal to zero 0

$$\frac{C_1C_2}{T_0}\int_0^{T_0} \cos(n+m)\omega_0 t\, dt = 0$$

the second integral is

$$\frac{C_1C_2}{T_0}\int_0^{T_0} \cos(n-m)\omega_0 t\, dt \begin{cases} = 0 & \text{for } n \neq m \\ = C_1C_2 & \text{for } n = m \end{cases}$$

~~Therefor the Power of~~

$g(t) = C_1 \cos n\omega_0 t + C_2 \cos m\omega_0 t$

$$P_{av} = \frac{C_1^2}{2} + \frac{C_2^2}{2} \quad \text{for } n \neq m$$

$$P_{av} = \frac{C_1^2}{2} + \frac{C_2^2}{2} + C_1 C_2 \quad \text{for } n = m$$

(c) $g(t) = D e^{j\omega_0 t}$

in this case the signal is complex and periodic, therefor we use eq. (1-6) to comput the power

$$P_{av} = \frac{1}{T_0} \int_0^{T_0} |g(t)|^2 dt = \frac{1}{T_0} \int_0^{T_0} D^2 |e^{j\omega_0 t}|^2 dt$$

not that $|e^{j\omega_0 t}| = 1$

$$\therefore P_{av} = \frac{1}{T_0} \int_0^{T_0} D^2 dt = \frac{1}{T_0} D^2 t \Big|_0^{T_0} = \frac{1}{T_0} D^2 T_0 = D^2$$

$$\therefore P_{av} = D^2$$

Comment: In Part (b) we showed that the Power of the sum of two sinusoids $g_1(t)$ and $g_2(t)$ having unequal frequency and they are multiple of a common frequency is equal to the sum of the powers of the sinusoids, that is

$$P_{av} = P_{av_1} + P_{av_2} - -$$

Ex. 1.12.

(a) Find the energy of the signal $g_1(t)$ shown in Fig.(1-28a)

(b) Find the energy of the signal $g_2(t)$ shown in Fig.(1-28b)

(c) Find the energy of the signal $g_3(t)$ shown in Fig.(1-28c)

(d) Find the energy of the signal $g_4(t)$ shown in Fig.(1-28d)

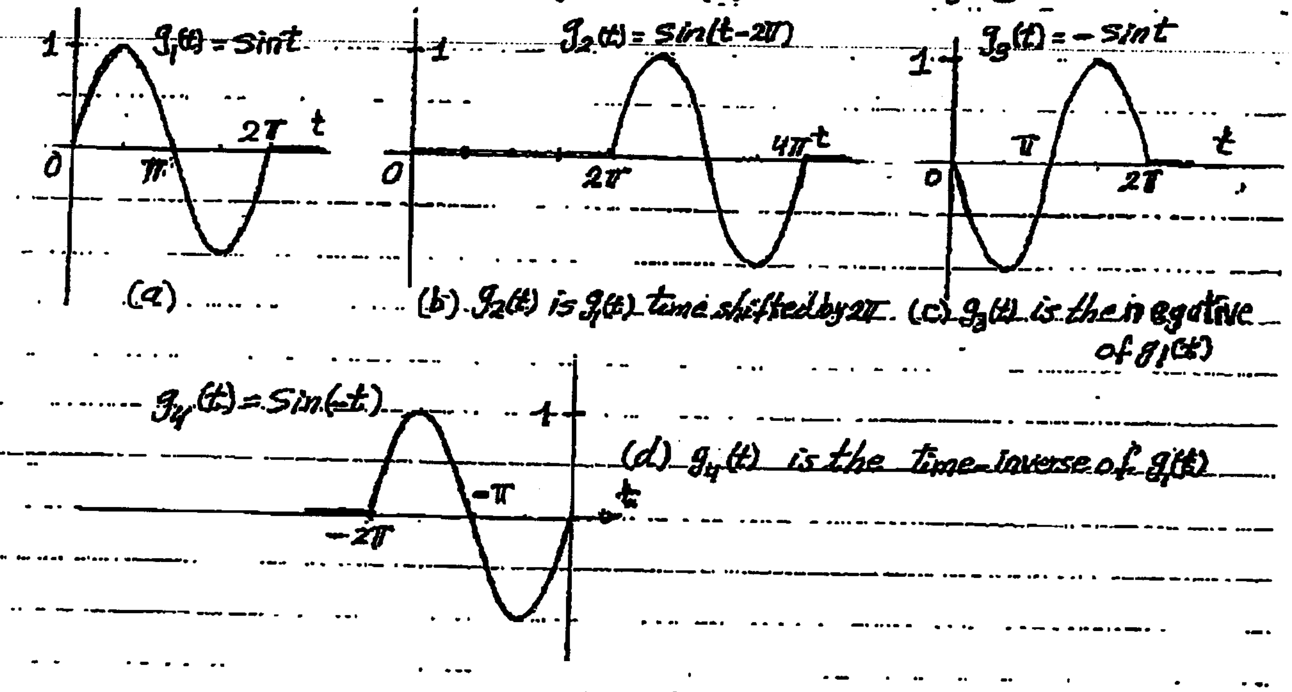


Fig.(1.28)

Solution :

(a) $g_1(t)$ is periodic signal but does not extends from $t=-\infty$ to $t=\infty$ but from $t=0$ to $t=2\pi$. Therefor it is an energy signal.

$$E_1 = \int_0^{T_0} |g_1(t)|^2\,dt = \int_0^{2\pi} \sin^2 t\,dt = \int_0^{2\pi} \frac{1}{2}(1-\cos 2t)\,dt = \frac{1}{2}\left[t - \frac{1}{2}\sin 2t\right]_0^{2\pi}$$

$$E_1 = \frac{1}{2}\left[2\pi - \frac{1}{2}\cancel{\sin 4\pi} - \left(0 - \frac{1}{2}\cancel{\sin 0}\right)\right] = \frac{1}{2}\,2\pi = \boxed{\pi}$$

(b)

$g_2(t)\ \sin(t-2\pi) \qquad 2\pi \le t \le 4\pi$

$g_2(t)$ is periodic but it is time limitted, therefor it is an energy signal. It is time-shifted by 2π to the right of $g_1(t)$

$$E_2 = \int_{t_0}^{T_0+t_0} |g_2(t)|^2\,dt = \int_{2\pi}^{4\pi} \sin^2(t-2\pi)\,dt$$

Since $\sin(t-2\pi) = \sin t$

$$\therefore\ E_2 = \int_{2\pi}^{4\pi} \sin^2 t\,dt = \boxed{\pi}$$

(c) $g_3(t) = -\sin t$ is the negative of $g_1(t)$. The signal is time limitted, i.e, $0 \le t \le 2\pi$, $T_0 = 2\pi$

$$E_3 = \int_0^{T_0} |g_3(t)|^2 dt = \int_0^{2\pi} (-\sin t)^2 dt = \int_0^{2\pi} \sin^2 t\,dt = \boxed{\pi}$$

(d) $g_4(t) = \sin(-t) \qquad -2\pi \le t \le 0$

is the time-inverse of $g_1(t)$

$$E_4 = \int |g_4(t)|^2 dt = \int_{-2\pi}^{0} (\sin(-t))^2 dt$$

Since $\sin(-t) = -\sin t$

$$\therefore\ E_4 = \int_{-2\pi}^{0} (-\sin t)^2 dt = \int_{-2\pi}^{0} \sin^2 t\,dt = \frac{1}{2}\left(t - \frac{\sin 2t}{2}\right)\Big|_{-2\pi}^{0} = \boxed{\pi}$$

~~Comment: The results show the Time-Shifting, negation~~ and time-invers have no effect on the energy of the signal.

Exa. 1.13

(a) Find the energy E_1 and E_2 of the signal pair $g_1(t)$ and $g_2(t)$ shown in Fig. (1-29a). Detetermine the area under $g_1(t) \cdot g_2(t)$ (by integration over its interval 0 to 2) and the energy of $g_1(t) + g_2(t)$.

(b) Repeat the procedure (a) for the signal pair shown in Fig.(1-29b)

(c) Repeat the procedure (a) for the signal pair shown in Fig.(1-29c). Why $E_{12} \neq E_1 + E_2$ and the area under $g_1(t) \cdot g_2(t) \neq 0$ in this case.

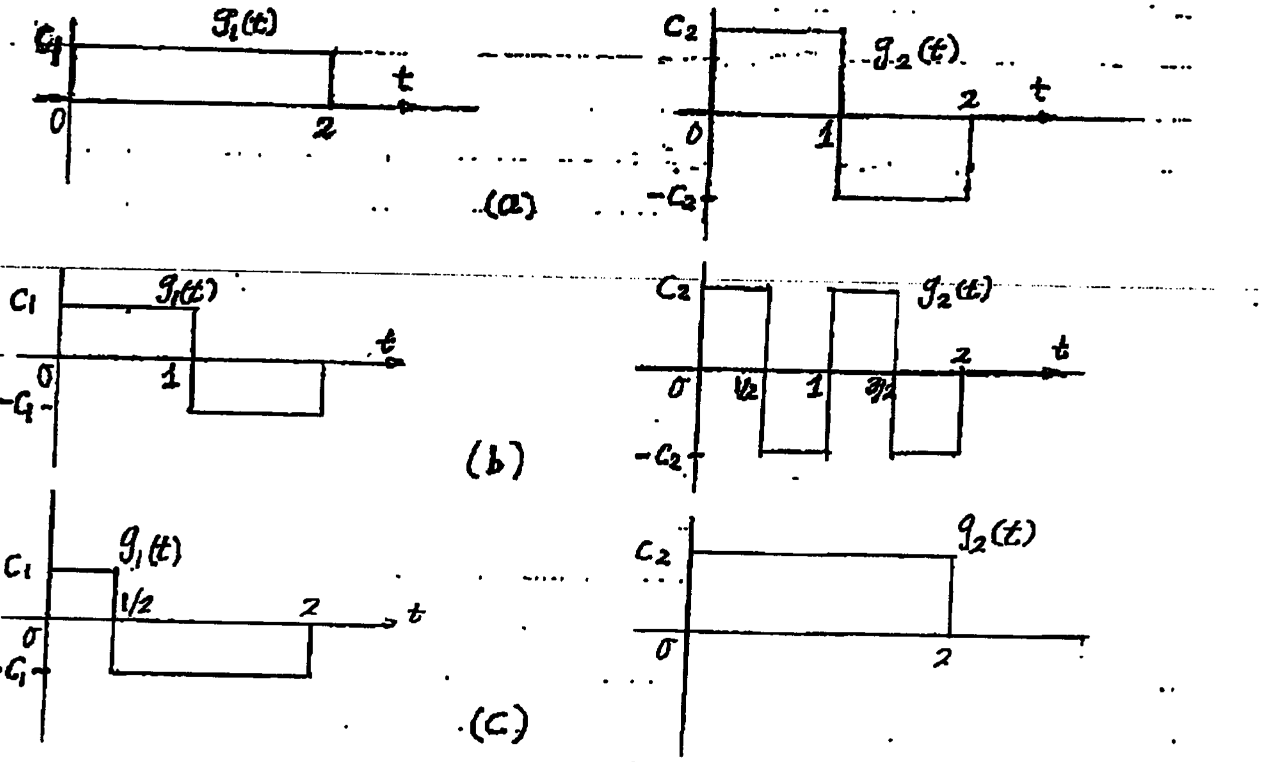


Fig. (1-29)

(C)

$$E_1 = \int_{-\infty}^{\infty} |g_1(t)|^2 dt = \int_0^2 g_1^2(t)\, dt = \int_0^2 C_1^2 dt = 2C_1^2$$

$$E_2 = \int_{-\infty}^{\infty} |g_2(t)|^2 dt = \int_0^2 g_2^2(t)\, dt = \int_0^2 C_2^2 dt = 2C_2^2$$

Area under $g_1(t) \cdot g_2(t) = E_{1.2} = \int_{-\infty}^{\infty} g_1(t) \cdot g_2(t)\, dt$

$$E_{1.2} = \int_0^{1/2} C_1 C_2\, dt + \int_{1/2}^{2} -C_1 C_2\, dt = C_1 C_2\, t\Big|_0^{1/2} - C_1 C_2\, t\Big|_{1/2}^{2}$$

$$E_{1.2} = \frac{C_1 C_2}{2} - \frac{3}{2} C_1 C_2 = \underline{\underline{-C_1 C_2}} \neq 0$$

$$E_{1+2} = \int_{-\infty}^{\infty} \left(g_1(t) + g_2(t)\right)^2 dt = \int_0^2 \left(g_1^2(t) + g_2^2(t) + 2\, g_1(t) \cdot g_2(t) \right) dt$$

$$= 2C_1^2 + 2C_2^2 + 2(-C_1 C_2)$$

$$= 2C_1^2 + 2C_2^2 - 2C_1 C_2 \neq E_1 + E_2$$

The reason that $E_{1\cdot2} \neq 0$ and $E_{1+2} \neq E_1 + E_2$ in case (C) will be explaind later when we discuss the Fourier-Series and the Orthogonality function

Exa - 1-14

Find the power of the periodic signal $g(t)$ shown in Fig-(1-30)

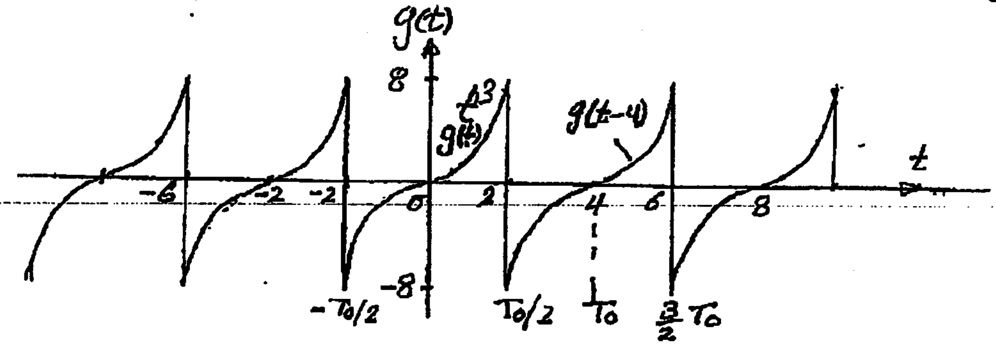


Fig-(1-30)

We use eq-(1-6) for calculating Pav.
we have to interat over one Period To, no matter from which Point to which point. We may integrat from 0 to To = 4 or from To/2 to 3/2 To or from -To/2 to To/2 where To = 4

we perform first the integration from To/2 = -2 to To/2=2

$$P_{av} = \frac{1}{T_o}\int_{-T_o/2}^{T_o/2} |g(t)|^2 dt = \frac{1}{4}\int_{-2}^{2} (t^3)^2 dt = \frac{1}{4}\int_{-2}^{2} t^6 dt$$

$$P_{av} = \frac{1}{4} \times \frac{1}{7} t^7 \Big|_{-2}^{2} = \frac{1}{4\times 7}\left(2^7 - (-2)^7\right) = \frac{1}{4\times 7}(128 + 128) = \frac{64}{7} \text{ wat}$$

$$P_{av} = \frac{64}{7} \text{ Watt}$$

If we want to integrat from 0 to To=4, then we have to integrat $|g(t)|^2$ from 0 to 2 and then integrat the time shift signal $g(t-4)$ from 2 to 4

~~where $g(t-4) = (t-4)^3$~~

$$\therefore P_{av} = \frac{1}{T_0}\left[\int_0^{T_0/2} |g(t)|^2 dt + \int_{T_0/2}^{T_0} |g(t-4)|^2 dt\right]$$

$$= \frac{1}{4}\left[\int_0^{2} (t^3)^2 dt + \int_2^4 ((t-4)^3)^2 dt\right]$$

$$= \frac{1}{4}\left[\int_0^{2} t^6 dt + \int_2^4 (t-4)^6 dt\right]$$

$$= \frac{1}{4}\left[\frac{t^7}{7}\Big|_0^2 + \frac{(t-4)^7}{7}\Big|_2^4\right]$$

$$= \frac{1}{7\times 4}\left[2^7 - 0 + (4-4)^7 - (2-4)^7\right]$$

$$= \frac{1}{4\times 7}(128 + 0 + 0 + 128) = \frac{64}{7} \text{ watt}$$

If we integrat from 2 to 6 then we have to integrate $g(t-4) = (t-4)^3$

$$P_{av} = \frac{1}{4}\int_2^6 (t-4)^6 dt = \frac{1}{4\times 7}(t-4)^7\Big|_2^6 = \frac{1}{4\times 7}\left[(6-4)^7 - (2-4)^7\right]$$

$$P_{av} = \frac{1}{4\times 7}(128 + 128) = \frac{64}{7} \text{ watt}$$

All the integrations give the same result as long we integrat over one Period(T_0) no matter from which Point to which Point and sabstitute the correct expression for the singnal..

1.4 Orthogonal Functions

~~The definition of the "Orthogonality" of function is~~ slitely different from the vector.

Two functions $\phi_1(t)$ and $\phi_2(t)$ are defined to be orthogonal over the interval (t_1, t_2) if

$$\int_{t_1}^{t_2} \phi_1(t) \cdot \phi_2(t)\, dt = 0 \qquad --- \quad (1-35)$$

In fact this integral gives the area the curve of the product of $\phi_1(t) . \phi_2(t)$ in the range t_1 to t_2

If $\phi_1(t)$ and $\phi_2(t)$ are complex function and are orthogonal over (t_1, t_2), then

$$\int_{t_1}^{t_2} \phi_1(t) \cdot \phi_2^*(t)\, dt = 0 \qquad --- \quad (1-36)$$

where ϕ^* means conjugate

Generaly, if $\phi_n(t)$ and $\phi_m(t)$ are members of set of Complex functions and are mutually orthogonal over the interval (t_1, t_2), then

$$\int_{t_1}^{t_2} \phi_n(t)\, \phi_m^*(t)\, dt = \begin{cases} 0 & \text{if } n \neq m \\ K_n & \text{if } n = m \end{cases} \qquad -- \quad (1-37)$$

The set of basis functions $\phi_n(t)$ $(\phi_1(t), \phi_2(t), \phi_3(t), ---)$ is

~~said to be "normalized" if~~

$$K_n = \int_{t_1}^{t_2} |\phi_n(t)|^2 dt = 1 \quad \text{for all } n \quad --- (4\text{-}38)$$

If the set is orthogonal and normalized, it is called an "Orthonormal set".

The integral of the product of two functions over a given interval is called the "Inner product" of the two functions.

Example 1.15 : Orthogonal functions

Prove that the following two functions are orthogonal

$\sin \pi t$ and $\sin 2\pi t$ over $[0, 2]$

Solution Since the function are real, then

$$\int_{t_1}^{t_2} \phi_1(t)\, \phi_2(t)\, dt = \int_0^2 \sin \pi t \, \sin 2\pi t \, dt$$

$$= \frac{1}{2}\left[\int_0^2 (\cos(\pi - 2\pi)t - \cos(\pi + 2\pi)t)\, dt\right]$$

$$= \frac{1}{2}\left[\int_0^2 \cos(-\pi t) - \cos 3\pi t)\, dt\right]$$

$$= \frac{1}{2}\left[\int_0^2 (\cos \pi t - \cos 3\pi t)\, dt\right] = \frac{1}{2}\left[\left(\frac{1}{\pi}\sin \pi t - \frac{1}{3\pi}\sin 3\pi t\right)\Big|_0^2\right]$$

5

$$\therefore \int_0^2 \sin \pi t \sin 2\pi t \, dt = \frac{1}{2}\left[\frac{1}{\pi}\sin 2\pi - \frac{1}{3\pi}\sin 6\pi\right] = 0$$

note that $\sin 2\pi = 0$ and $\sin 6\pi = 0$

Therefore the functions $\sin \pi t$ and $\sin 2\pi t$ are orthogonal over $(0, 2)$

Example 1.16

Show If the following set of functions are orthogonal over the given intervals.

(a) $\sin 2\omega_0 t$ and $\sin 3\omega_0 t$ $\left(\omega_0 = \frac{2\pi}{T_0}\right)$ over $(0, T_0)$

(b) $\sin 2\omega_0 t$ and $\sin 3\omega_0 t$ $\left(\omega_0 = \frac{2\pi}{T_0}\right)$ over $\left(\frac{T_0}{4}, \frac{5}{4}T_0\right)$

(c) $\sin 2\omega_0 t$ and $\sin 3\omega_0 t$ $\omega_0 = \frac{2\pi}{T_0}$ over $\left(\frac{T_0}{3}, \frac{4}{3}T_0\right)$

Solution

(a) $$\int_0^{T_0} \sin 2\omega_0 t \sin 3\omega_0 t \, dt = \frac{1}{2}\left[\int_0^{T_0} (\cos(-\omega_0 t) - \cos 5\omega_0 t) dt\right.$$

$$= \frac{1}{2}\int_0^{T_0} (\cos \omega_0 t - \cos 5\omega_0 t) dt = \frac{1}{2}\left[\left[\frac{1}{\omega_0}\sin \omega_0 t - \frac{1}{5\omega_0}\sin 5\omega_0 t\right)\Big|_0^{T_0}\right]$$

$$= \frac{1}{2}\left[\frac{1}{\omega_0}(\sin \omega_0 T_0 - \sin 0) - \frac{1}{5\omega_0}(\sin 5\omega_0 T_0 - \sin 0)\right]$$

$$= \frac{1}{2}\left[\frac{1}{\omega_0}(0 - 0) - \frac{1}{5\omega_0}(0 - 0)\right] = 0$$

~~Therefore the functions Sin2ω₀t and Sin3ω₀t are~~ Orthogonal over the interval $(0, T_0)$

(b)

$$\int_{T_0/4}^{5T_0/4} (\sin 2\omega_0 t \sin 3\omega_0 t)\,dt = \frac{1}{2}\left[\left(\frac{\sin\omega_0 t}{\omega_0} - \frac{\sin 5\omega_0 t}{5\omega_0}\right)\Big|_{T_0/4}^{5T_0/4}\right]$$

$$= \frac{1}{2}\left[\frac{1}{\omega_0}\left(\sin\tfrac{5}{4}\omega_0 T_0 - \sin\tfrac{\omega_0 T_0}{4}\right) - \frac{1}{5\omega_0}\left(\sin\tfrac{25}{4}\omega_0 T_0 - \sin\tfrac{5}{4}\omega_0 T_0\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{\omega_0}\left(\sin\tfrac{5}{2}\pi - \sin\tfrac{\pi}{2}\right) - \frac{1}{5\omega_0}\left(\sin\tfrac{25}{2}\pi - \sin\tfrac{5}{2}\pi\right)\right], \quad \underline{\text{Note}: \omega_0 T_0 = 2\pi}$$

$$\sin\tfrac{5}{2}\pi = \sin\left(2\pi + \tfrac{\pi}{2}\right) = \sin\pi/2$$

$$\sin\tfrac{25}{2}\pi = \sin\left(10\pi + \tfrac{5\pi}{2}\right) = \sin\tfrac{5}{2}\pi$$

$$\therefore \int_{T_0/4}^{5T_0/4} \cdots = \frac{1}{2}\left[\frac{1}{\omega_0}\left(\sin\tfrac{\pi}{2} - \sin\tfrac{\pi}{2}\right) - \frac{1}{5\omega_0}\left(\sin\tfrac{5}{2}\pi - \sin\tfrac{5}{2}\pi\right)\right]$$

$$= 0$$

$\therefore$ $\sin 2\omega_0 t$ and $\sin 3\omega_0 t$ are orthogonal over the interval $(T_0/4, 5T_0/4)$

(c)

$$\int_{T_0/3}^{4T_0/3} \sin 2\omega_0 t \sin 3\omega_0 t\,dt = \frac{1}{2}\left[\left(\frac{1}{\omega_0}\sin\omega_0 t - \frac{1}{5\omega_0}\sin 5\omega_0 t\right)\Big|_{T_0/3}^{4T_0/3}\right]$$

$$= \frac{1}{2}\left[\frac{1}{\omega_0}\left(\sin\tfrac{4}{3}\omega_0 T_0 - \sin\tfrac{\omega_0 T_0}{3}\right) - \frac{1}{5\omega_0}\left(\sin\tfrac{20}{3}\omega_0 T_0 - \sin\tfrac{5}{3}\omega_0 T_0\right)\right]$$

$$\int_{T_0/3}^{4T_0/3} \cdots = \frac{1}{2}\left[\frac{1}{\omega_0}\left(\sin\frac{8}{3}\pi - \sin\frac{2}{3}\pi\right) - \frac{1}{5\omega_0}\left(\sin\frac{40}{3}\pi - \sin\frac{10}{3}\pi\right)\right]$$

Note that, $\omega_0 T_0 = 2\pi$

since $\sin\frac{8}{3}\pi = \sin(2\pi + \frac{2}{3}\pi) = \sin\frac{2}{3}\pi$

$\sin\frac{40}{3}\pi = \sin(10\pi + \frac{10}{3}\pi) = \sin\frac{10}{3}\pi$

$$\therefore \int_{T_0/3}^{4T_0/3} \cdots = \frac{1}{2}\left[\frac{1}{\omega_0}\left(\sin\frac{2}{3}\pi - \sin\frac{2}{3}\pi\right) - \frac{1}{5\omega_0}\left(\sin\frac{10}{3}\pi - \sin\frac{10}{3}\pi\right)\right]$$

$$= 0$$

$\therefore$ $\sin 2\omega_0 t$ and $\sin 3\omega_0 t$ are orthogonal over the interval $(T_0/3, 4T_0/3)$

This example shows that the set of periodic trigonometric functions whose frequencies are integers are always orthogonal over any interval of length of one period no matter from which point to which point. It may be from t_1 to $T_0 + t_1$.

Hence
$$\int_{t_1}^{T_0+t_1} \sin n\omega_0 t \, \sin m\omega_0 t \, dt = \begin{cases} 0 & \text{for } n \neq m \\ K_n & \text{for } n = o \end{cases} \qquad (1\text{-}39a)$$

$$\int_{t_1}^{T_0+t_1} \cos n\omega_0 t \, \cos m\omega_0 t \, dt = \begin{cases} 0 & \text{for } m \neq m \\ K_n & \text{for } n = m \end{cases} \qquad (1\text{-}39b)$$

$$\int_{t_1}^{T_0+t_1} \sin n\omega_0 t \, \cos m\omega_0 t \, dt = 0 \quad \text{for all } n \text{ and } m \qquad (1\text{-}39c)$$

~~Complete Set : Now We introduce the concept~~ of "Complete set" - A Complete set of functions is defined as

"The set that consists of all functions Which are Orthogonal to each Other in a given intervl"

As an example the trigonometric functions Over one Period (t_1, T_0+t_1)

The complete trigonometric function cosists of infinit Number of orthagonal functions over (t_1, T_0+t_1) where T_0 is the Periode of the fundamental

Exponetial functions: Anothe set of orthogonal functions is the exponential $e^{jn\omega_0 t}$ $(n = 0, \pm 1, \pm 2, ---)$ over any interval of duration $T_0 = 2\pi/\omega_0$, that is

$$\int_{t_1}^{T_0+t_1} e^{jn\omega_0 t} (e^{jm\omega_0 t})^* dt = \int e^{j(n-m)\omega_0 t} dt = \begin{cases} 0 & \text{for } n \neq m \\ T_0 & n = 0 \end{cases} \quad \text{--- (1-40)}$$

Trigonometric Fourier Series

Any signal waveform theoretically, consists of infinite number of harmonics of sinusoid and the frequency of each harmonic is a multiple of a certain harmonic called the fundamental. Therefor the waveform is a supperposition of sinusoid function. This is in fact the idea of "Fourier series" which mathematicaly states that any signal waveform $f(t)$ can be represented by complete trigonometric set over an interval of T_0, that is

$$f_F(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \qquad -\infty < t < \infty \qquad (1.42)$$

where $f_F(t)$ means the Fourier series of $f(t)$

$\omega_0 = 2\pi/T_0$ The fundmental frequency.

a_0, a_n and b_n are called the Fourier coefficients and can be determined using the following eqs.

$$a_0 = \frac{1}{T_0}\int_{t_1}^{t_1+T_0} f(t)\,dt \qquad (1.43a)$$

$$a_n = \frac{2}{T_0}\int_{t_1}^{t_1+T_0} f(t)\cos n\omega_0 t\,dt \qquad n = 1,2,3,\ldots \qquad (1.43b)$$

$$b_n = \frac{2}{T_0}\int_{t_1}^{t_1+T_0} f(t)\sin n\omega_0 t\,dt \qquad n = 1,2,3,\ldots \qquad (1.43c)$$

~~Note that $f(t)$ is defined over the interval $t_1 \le t \le t_1 + T_0$~~
that is

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \qquad t_1 \le t \le t_1 + T_0 \quad \text{---} (1.42)$$

. The only difference between eq. (1.42) and eq. (1.44) is in the interval of existence. $f_T(t)$ exists over $-\infty < t < \infty$ whereas $f(t)$ exists over $t_1 \le t \le t_1 + T_0$.

Compact Trigonometric Fourier Series

For drawing the Amplitude of each harmonic against frequency, it is necessary to combine (Sine) and (Cosine) terms with the same frequency in a single term using the following Identity

$$a_n \cos n\omega_0 t + b_n \sin n\omega_0 t = C_n \cos(n\omega_0 t + \theta)$$

$$\text{Where } C_n = \sqrt{a_n^2 + b_n^2} \qquad \text{------} (1.43a)$$

$$\theta_n = \tan^{-1} \frac{-b_n}{a_n} \qquad \text{--------} (1.43b)$$

$$\text{and for } n = 0 \Longrightarrow C_0 = a_0 \qquad \text{----} (1.43c)$$

Therefore the Trigonometric Fourier Series in Compact form is

$$f_T(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n) \qquad \text{----} (1.44)$$

~~and $f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$ over $t_1 \le t \le t_1 + T_0$ --- (1.45)~~

Note that C_n represents the amplitude spectrum and θ_n the phase spectrum. See that there is no differene between Eqs. 1.44 and 1.45 only in the interval where they exists. f.r. is periodic

Ex.- 1.20

Find the compact trigonometric Fourier series for the exponential $f(t) = e^{-t/2}$ shown in Fig (1.36) over $0 < t < \pi$

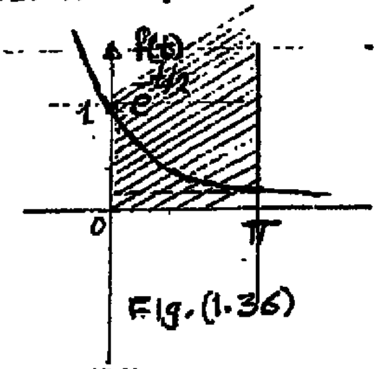


Fig. (1.36)

Solution

Since the interval is $0 \le t \le \pi$, therefore $T_0 = \pi$ and the fundametal frequency is

$$\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{\pi} = 2 \text{ rad/sec}$$

and $f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos 2nt + b_n \sin 2nt)$ over. $0 \le t \le \pi$

where

$$\underline{a_0} = \frac{1}{T_0} \int_0^{T_0} f(t)\,dt = \frac{1}{\pi} \int_0^{\pi} e^{-t/2}\,dt = -\frac{1}{2\pi} e^{-t/2} \Big|_0^{\pi} = \underline{0.504}$$

$$a_n = \frac{2}{T_0} \int_0^{T_0} f(t) \cos n\omega_0 t = \frac{2}{\pi} \int_0^{\pi} e^{-t/2} \cos 2nt\,dt$$

$$\int_0^{\pi} e^{-t/2} \cos 2nt\,dt = -2e^{-t/2} \cos 2nt \Big|_0^{\pi} - \int_0^{\pi} -2e^{-t/2} (-2n \sin 2nt)\,dt$$

$$= -2(e^{-\pi/2} - 1) - 4n \int_0^{\pi} e^{-t/2} \sin 2nt\,dt = 1.5842 - 4n \int_0^{\pi} e^{-t/2} \sin 2nt\,dt$$

$$\cancel{\int_0^{\pi} e^{-t/2} \cos 2nt\,dt = 1.5842 - 4n \int_0^{\pi} e^{-t/2} \sin 2nt\,dt}$$

$$= 1.5842 - 4n\left[-2e^{-t/2} \sin 2nt \Big|_0^{\pi} - \int_0^{\pi} -2e^{-t/2}(2n\cos 2nt)\,dt\right]$$

$$\therefore \int_0^{\pi} e^{-t/2} \cos 2nt\,dt = 1.5842 - 16n^2 \int_0^{\pi} e^{-t/2} \cos 2nt\,dt$$

$$(1+16n^2) \int_0^{\pi} e^{-t/2} \cos 2nt\,dt = 1.5842$$

$$\therefore \int_0^{\pi} e^{-t/2} \cos 2nt\,dt = \frac{1.5842}{1+16n^2}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} e^{-t/2} \cos 2nt\,dt = \frac{1.5842}{\pi} \cdot \frac{2}{1+16n^2} = 0.504 \cdot \frac{2}{1+16n^2}$$

In similar way, we find

$$b_n = \frac{2}{T_0} \int_0^{T_0} f(t) \cos n\omega_0 t\,dt = \frac{2}{\pi} \int_0^{\pi} e^{-t/2} \cos 2nt\,dt = 0.504 \cdot \frac{8}{1+16n^2}$$

Therefore $f(t) = 0.504\left(1 + \sum_{n=1}^{\infty} \frac{2}{1+16n^2}(\cos 2nt + 4n \sin 2nt)\right)$ $\quad 0 \le t \le \pi$

Note that $f(t)$ is defined over the interval $0 \le t \le \pi$ only whereas $f_r(t)$ the Fourier series is defined over the interval $-\infty < t < \infty$. The Fourier series is periodical waveform.

Using 11 terms the Fourier Series canbe Written as

$$f(t) = 0.504\Big[1 + \frac{2}{17}(\cos 2t + 4\sin 2t) + \frac{2}{65}(\cos 4t + 8\sin 4t) + \frac{2}{145}(\cos 6t + 12\sin 6t)$$

$$+ \frac{2}{257}(\cos 8t + 16\sin 8t) + \frac{2}{401}(\cos 10t + 20\sin 10t) + \frac{2}{577}(\cos 12t + 24\sin 12t)$$

$$+ \frac{2}{785}(\cos 14t + 28\sin 14t) + \frac{2}{1025}(\cos 16t + 32\sin 16t) + \frac{2}{1296}(\cos 18t + 36\sin 18t)$$

$$+ \frac{2}{1601}(\cos 20t + 40\sin 40t)\Big]$$

Insert for $t = \pi/4, \pi/2, 3\pi/4$ in the above expression, we find

$$f(\pi/4) = 0.504\Big[1 + \frac{2}{17}(\cos \pi/2 + 4\sin \pi/2) + \frac{2}{65}(\cos \pi + 8\sin \pi) + \frac{2}{145}(\cos 6\pi/4 + 12\sin 6\pi/4)$$

$$+ \frac{2}{257}(\cos 2\pi + 16\sin 2\pi) + \frac{2}{401}(\cos 10\pi/4 + 20\sin 10\pi/4) + \frac{2}{577}(\cos 12\pi/4 + 24\sin 12\pi/4)$$

$$+ \frac{2}{785}(\cos 14\pi/4 + 28\sin 14\pi/4) + \frac{2}{1025}(\cos 16\pi/4 + 24\sin 16\pi/4)$$

$$+ \frac{2}{1296}(\cos 18\pi/4 + 36\sin 18\pi/4) + \frac{2}{1601}(\cos 20\pi/4 + 40\sin 20\pi/4) + \frac{2}{1937}(\cos 22\pi/4 + 44\sin 22\pi/4)\Big]$$

$$f(\pi/4) = .504\Big[1 + \frac{8}{17} - \frac{2}{65} - \frac{24}{145} + \frac{2}{257} + \frac{40}{401} - \frac{2}{577} - \frac{56}{785} + \frac{2}{1025} + \frac{72}{1296} - \frac{2}{1601} - \frac{88}{1937}\Big]$$

$$= 0.6642$$

Is similar way

$$f(\pi/2) = 0.504\Big[1 - \frac{2}{17} + \frac{2}{65} - \frac{1}{145} + \frac{2}{257} - \frac{2}{401} + \frac{2}{577} - \frac{2}{785} + \frac{2}{1025} + \frac{2}{1296} + \frac{2}{1601} - \frac{2}{1937}\Big]$$

$$= 0.4554$$

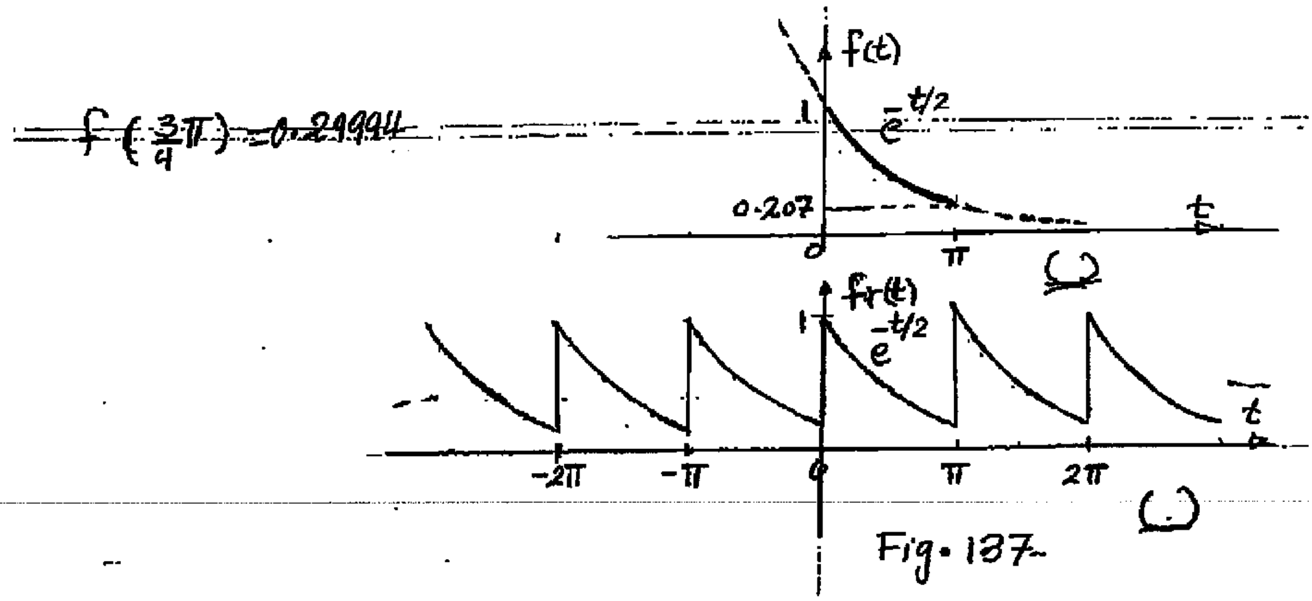


Fig. 137

Fig. (1.37a) is the function $f(t)$ which extents from $t=0$ to $t=\pi$ whereas Fig. (1.37b) represents the Fourier series which is periodic function

Now we will express the Fourier series using the compact Trigonometris representaion. We have found previously

$$a_0 = 0.504 \quad ; \quad a_n = 0.504\left(\frac{2}{1+16n^2}\right), \quad b_n = 0.504\left(\frac{8n}{1+16n^2}\right)$$

The Fourier Coefficcents using eqs. (1.43)

$$C_0 = a_0 = 0.504$$

$$C_n = \sqrt{a_n^2 + b_n^2} = 0.504\sqrt{\frac{4}{(1+16n^2)^2} + \frac{64n^2}{(1+16n^2)^2}} = 0.504\left(\frac{2}{\sqrt{1+16n^2}}\right)$$

$$\theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right) = \tan^{-1}(-4n) = -\tan^{-1} 4n$$

The amplitudes and phases of the (dc) and the first seven

~~harmonics are computed using the forgoing eqs. and displayed~~ in the table below

n	0	1	2	3	4	5	6	7
C_n	0.504	0.244	0.125	0.084	0.063	0.0504	0.042	0.036
θ_n	0	-75.96	-82.87	-85.24	-86.42	-87.14	-87.61	-87.95

Using these values, we can express $f(t)$ in the compact form

$$f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n) \qquad t_1 \le t \le t_2$$

concerning our example with $\omega_0 = 2$ rad/sec

$$f(t) = C_0 + C_1 \cos(2t + \theta_1) + C_2 \cos(4t + \theta_2) + C_3 \cos(6t + \theta_3) + C_4 \cos(8t + \theta_4)$$
$$+ C_5 \cos(10t + \theta_5) + C_6 \cos(12t + \theta_6) + C_7 \cos(14t + \theta_8)$$

$$f(t) = 0.504 + 0.244 \cos(2t - 75.96°) + 0.125 \cos(4t - 82.87°) + 0.084 \cos(6t - 85.24°)$$

$$+ 0.063 \cos(8t - 86.42°) + 0.0504 \cos(10t - 87.14°) + 0.042 \cos(12t - 87.61°)$$

$$+ 0.036 \cos(14t - 87.95°) \qquad 0 \le t \le \pi$$

The Fourier Spectra (amplitude and phase against frequency) C_n and θ_n are drawn in Fig (1.38) using the value given in the table above.

Note that $\omega = n\omega_0 = 2n$

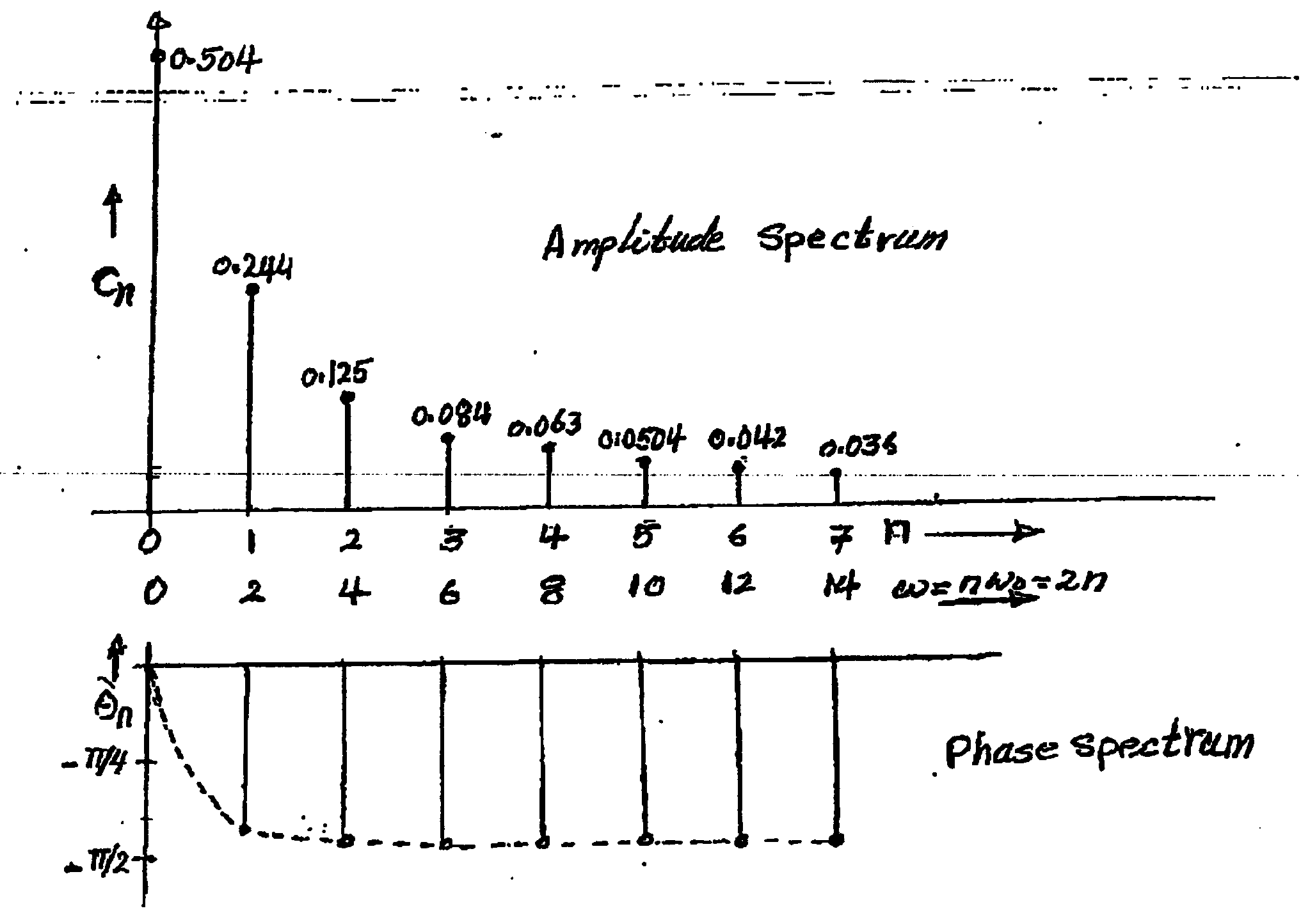


Fig. (1-3.8) Fourier spectra

Summary: We have shown how an arbitrary signal $f(t)$ may be expressed as a trigonometric Fourier series over any interval of $T_0 = (t_2 - t_1)$ seconds. The Fourier series is equal to $f(t)$ over this interval alone. Outside this interval the Fourier series is not necessarily equal to $f(t)$. The trigonometric Fourier series is a periodic function of period T_0 (The period of the fundamental).

The compact trigonometric Fourier series indicates that any function $f(t)$ (periodic or nonperiodic) can be expressed as a sum of sinusoids of frequencies 0 (dc), ω_0, $2\omega_0$, - - -, $n\omega_0$, whose amplitudes are C_0, C_1, C_2, - - -, C_n - - - and whose phases are

$\dots, \theta, \theta_1, \theta_2, \dots, \theta_n, \dots$

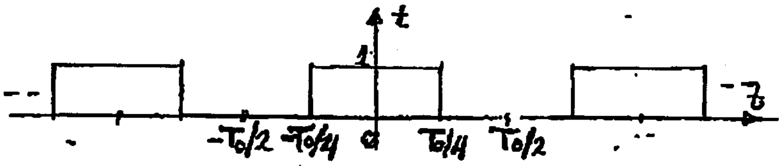
The plots of amplitude C_n vs. ω (amplitude spectrum) and θ_n vs. ω (Phas spectrum) are called the frequency spectra of $f(t)$.

Another example where the signal to be analysed is aperiodic.

Exa. 1.21

Find the compact trigonometric Fourier series for the periodic square wave $f(t)$ shown in Fig.(1-39) and sketch its amplitude and phase spectra

Solution



The Fourier series is

Fig.(1.39)

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

with $\omega_0 = 2\pi / T_0$

where $a_0 = \frac{1}{T_0} \int_{-T_0/4}^{T_0/4} dt = \frac{1}{2}$

$$a_n = \frac{2}{T_0} \int_{-T_0/4}^{T_0/4} \cos n\omega_0 t \, dt = \frac{2}{n\pi} \sin \frac{n\pi}{2}$$

$$a_n = \begin{cases} 0 & n \text{ even} \\ \frac{2}{n\pi} & n = 1, 5, 9, 13, \dots \\ -\frac{2}{n\pi} & n = 3, 7, 11, 15, \dots \end{cases}$$

$$b_n = \frac{2}{T_0}\int_{-T_0/4}^{T_0/4} \sin n\omega_0 t \, dt = 0$$

Therefore

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\left(\cos\omega_0 t - \frac{1}{3}\cos 3\omega_0 t + \frac{1}{5}\cos 5\omega_0 t - \frac{1}{7}\cos 7\omega_0 t + \cdots\right)$$

Note that $b_n = 0$ and all the (sine) terms are zero. Only the (Cosine) terms appear in the trigonometric series. Since there is only one coefficient in the series, therefore there is two alternatives for ploting the spectra

1- Ploting a_n where the phase is included in the sign of the coefficients as shown in Fig.-(1-40)

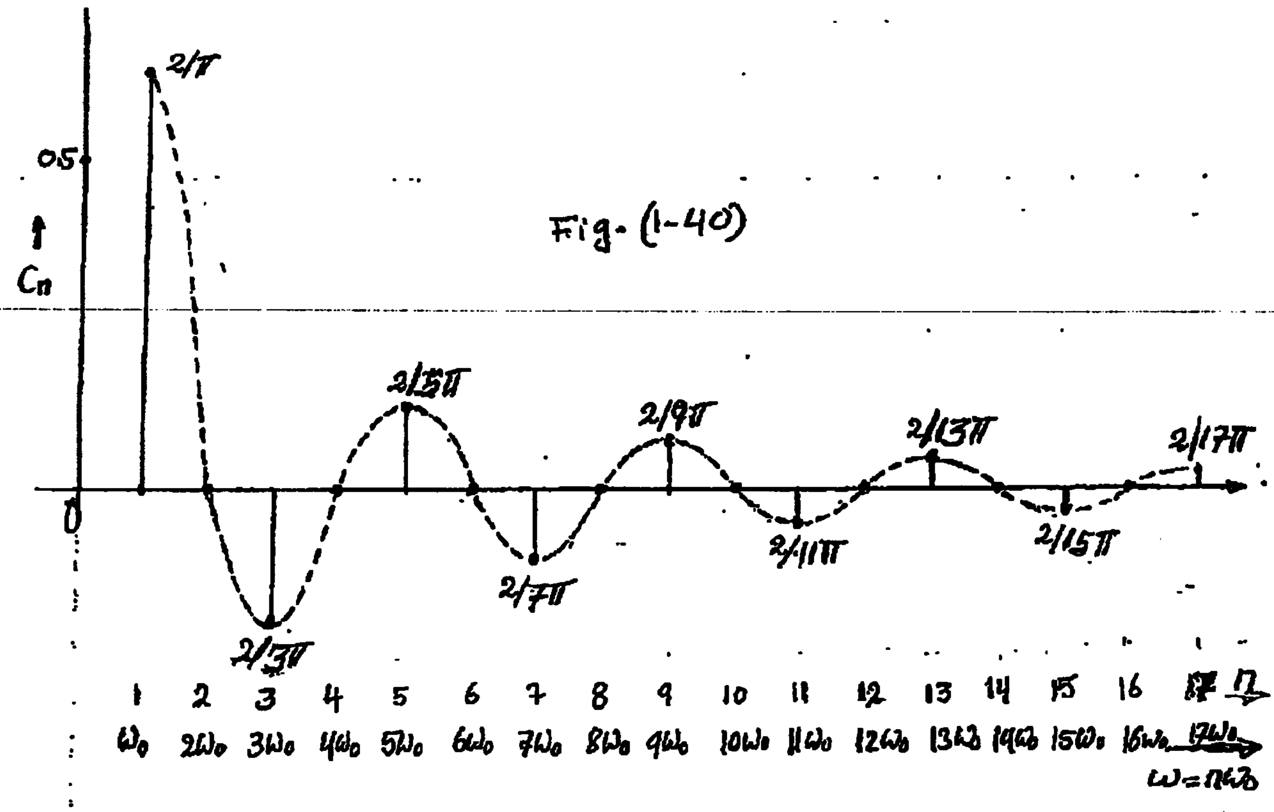


Fig.-(1-40)

~~2- Since amplitudes C_n by definition (eq. (1.43a)) are positive,~~ the negative sign can be represented by a phase of π radians. This can be seen from the identity

$$-\cos x = \cos(x - \pi).$$

Using this fact, the series can be expressed as

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\Big[\cos\omega_0 t + \frac{1}{3}\cos(3\omega_0 t - \pi) + \frac{1}{5}\cos(5\omega_0 t - \pi)$$

$$+ \frac{1}{7}\cos(7\omega_0 t - \pi) + \frac{1}{9}\cos(9\omega_0 t - \pi) + \cdots \cos(11\omega_0 t - \pi) - \frac{1}{13}\cos(13\omega_0 t - \pi)\Big]$$

This is the desired form of the compact trigonometric Fourier series. The amplitudes are

$$C_0 = \frac{1}{2}$$

$$C_n = \begin{cases} 0 & n \text{ even} \\ \dfrac{2}{n\pi} & n \text{ odd} \end{cases}$$

$$\theta_n = \begin{cases} -\pi & \text{for } n = 3, 7, 11, 15, \ldots \\ 0 & \text{for all } n \neq 3, 7, 11, 15, \ldots \end{cases}$$

The plots of the amplitudes and phase are shown in Fig. (1.41)

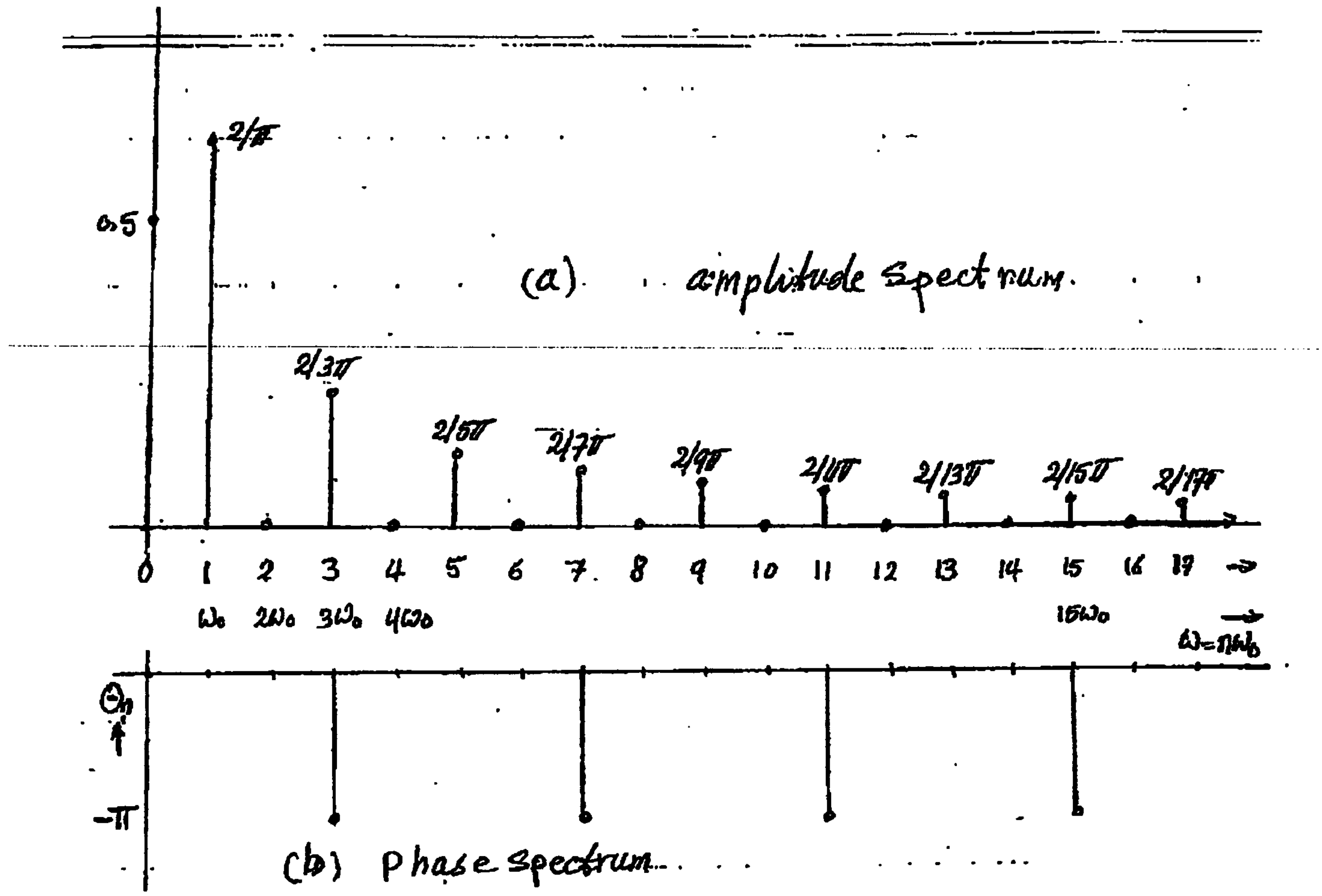


Fig (1-41)

Note : The first methode is the most convenient methode and we will use it in all our discussions.

~~Examp. 1.22 : Find the compact trigonometric Fourie Series~~ for the periodic square wave $f(t)$ shown in Fig (1.42) and sketch its amplitude and phase spectra.

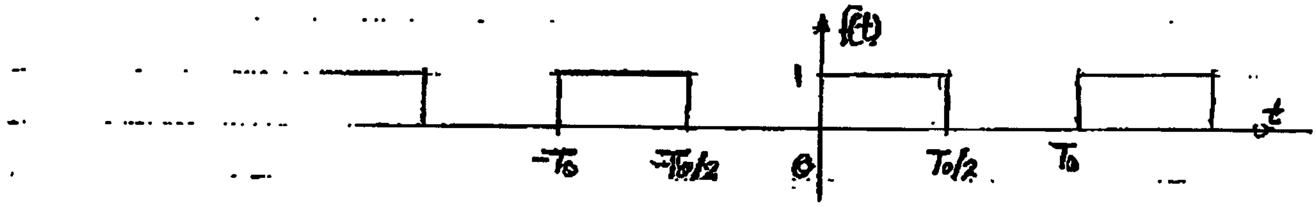


Fig-(1.42)

Solution

$$a_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f(t)\,dt = \frac{1}{T_0}\int_{0}^{T_0/2} dt = \frac{1}{2}$$

$$a_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} f(t)\cos n\omega_0 t\,dt = \frac{2}{T_0}\int_{0}^{T_0/2} \cos n\omega_0 t\,dt = \frac{2}{T}\cdot\frac{1}{n\omega_0}(\sin n\omega_0 t)\Big|_{0}^{T_0/2}$$

$$a_n = 0$$

$$b_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} f(t)\sin n\omega_0 t\,dt = \frac{2}{T_0}\int_{0}^{T_0/2} \sin n\omega_0 t\,dt = \frac{2}{T_0}\cdot\frac{1}{n\omega_0}(-\cos n\omega_0 t)\Big|_{0}^{T_0/2}$$

since $\omega_0 T_0 = 2\pi$

$$\therefore\ b_n = -\frac{2}{2n\pi}(\cos n\pi - 1) = -\frac{1}{n\pi}(\cos n\pi - 1)$$

$$b_n = \begin{cases} 0 & n \text{ even} \qquad \text{since } \cos n\pi = 1 \\ \dfrac{2}{n\pi} & n \text{ odd} \qquad \text{since } \cos n\pi = -1 \end{cases}$$

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\left(\sin\omega_0 t + \frac{1}{3}\sin 3\omega_0 t + \frac{1}{5}\sin 5\omega_0 t + \frac{1}{7}\sin 7\omega_0 t + \frac{1}{9}\sin 9\omega_0 t + \frac{1}{11}\sin 11\omega_0 t + \cdots\right)$$

The amplitude spectrum of Fourier series is shown in Fig (1.43)

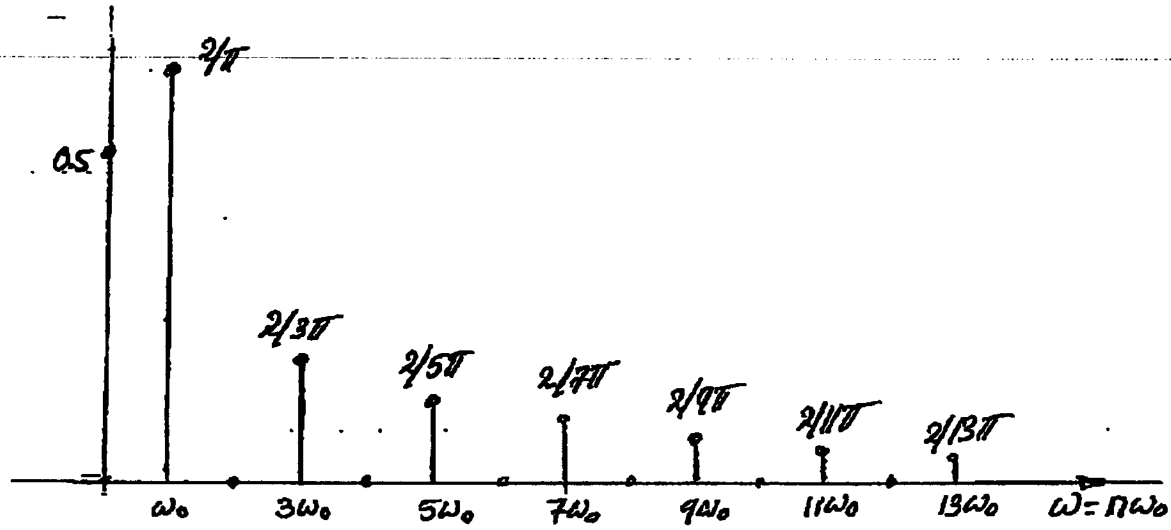


Fig. (1.43)

Symmetry Signal : The following Notes is for Periodic Signal

1- if $f(t) = f(-t)$, the signal is called even-symmetry as in fig.(1.39) Sample (1.21) p. 83

2- if $f(t) = -f(-t)$ the signal is called odd symmetry as in fig.(1-42) example (1.22) p. 87

3- If the signal neither eveven symmetry nor odd symmetry then the signal is not symmetric as in fig.(1.36) example (1-20) p. 77

4- If the signal is not symmetric, then the Fourier Series contains (Cosin terms as Well (Sine) terms as in exa. 1.20

5- If the signal is even, then there is only the (cosine) terms a_n whereas the sine-terms (b_n) are Zero as in example 1.21

6- If the signal is odd, then there is only the Sine-terms b_n whereas the cosine-terms a_n are Zero as in example 1.22

7- If the signal is symmetry, the Fourier Coefficients (a_n, b_n) can be evaluated by integrating the periodic signal over half-cycle only multiplied by 2, that is

$$\int_{-T_0/2}^{T_0/2} f(t)\,dt = 2\int_{-T_0/2}^{T_0/2} f(t)\,dt$$

8- ~~If $f_e(t)$ and $f_o(t)$ are even and odd functions respectively,~~ of t, then

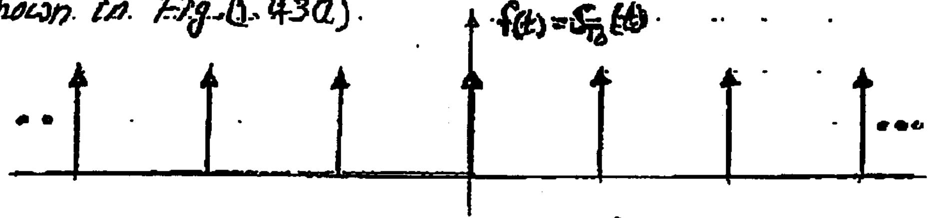
$$\int_{-a}^{a} f_e(t)\,dt = 2\int_{0}^{a} f_e(t) \quad \text{and} \quad \int_{-a}^{a} f_o(t)\,dt = 0$$

9- The product of an even and odd functions is an <u>odd</u> function

The product of two odd functions is an even function

The product of two even functions is an even function

~~Ex 1.23: Find the trigonometric Fourier series and sketch~~ the corresponding spectra for the periodic impulse function shown in Fig.(1.43a).

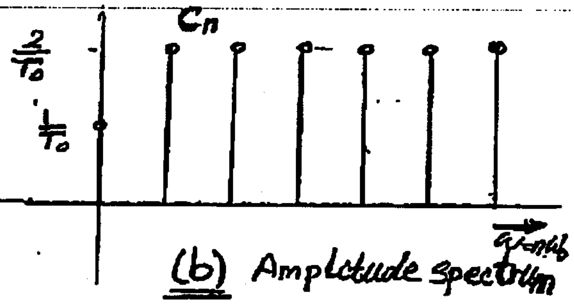


(a)

Solution

The trigonometric Fourier series for $\delta_{T_0}(t)$ is given by (see eq.1.44)

$$\delta_{T_0}(t) = C_0 + \sum C_n \cos(n\omega_0 t + \theta_0) .$$



(b) Amplitude spectrum

Fig.(1.43)

We first compute a_0, a_n and b_n

$$a_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t)\,dt = \frac{1}{T_0} \qquad \text{since} \int_a^b \delta(t)\,dt = 1 \quad \text{see eq.(1.15b)} \quad 0<a<b$$

$$a_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t)\cos n\omega_0 t\,dt = \frac{2}{T_0}\cos n\omega_0 t\Big|_{t=0} = \frac{2}{T_0} \qquad \text{see eq(1.19a)}$$

$$b_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t)\sin(n\omega_0 t)\,dt = \frac{2}{T_0}\sin(n\omega_0 t)\Big|_{t=0} = 0 \qquad \text{see eq.(1.19a)}$$

Therefore

$$C_0 = a_0 = \frac{1}{T_0}, \quad C_n = \sqrt{a_n^2+b_n^2} = \frac{2}{T_0}, \quad \theta_n = \tan^{-1}\frac{-b_n}{a_n} = \tan^{-1} 0 = 0$$

Thus

$$\delta_{T_0}(t) = \frac{1}{T_0}\left(1 + 2\sum_{n=1}^{\infty} \cos n\omega_0 t\right)$$

$$= \frac{1}{T_0}(1 + 2(\cos\omega_0 t + \cos 2\omega_0 t + \cos 3\omega_0 t + \cos 4\omega_0 t + \text{-----})$$

Fig(1.43b) shows the amplitude spectrum. The phase spectrum is zero

Exponential Fourier Series

- The most important and convenient representation of Fourier Series is by using a set of exponentials

$e^{jn\omega_0 t}$ $(n = 0, \pm 1, \pm 2, \pm 3, ---)$

Two complex functions $\phi_n(t)$, $\phi_m(t)$ are orthogonal over (t_1, t_2) if

$$\int_{t_1}^{t_2} \phi_n(t)\, \phi_m^*(t)\, dt = \begin{cases} \text{Constant} & \text{for } m = n \\ 0 & \text{for } m \neq n \end{cases} \quad --- (1.46)$$

The complexed exponential set $e^{-jn\omega_0 t}$ is a complete orthogonal set over any interval of duaration $T_0 = 2\pi/\omega_0$, that is

$$\int_{t_1}^{t_1+T_0} e^{jn\omega_0 t} \left(e^{jm\omega_0 t}\right)^* dt = \begin{cases} 0 & \text{for } m \neq n \\ T_0 & \text{for } m = n \end{cases} \quad --- (1.47)$$

The signal $f(t)$ can be expressed over an interval of duaration T_0 as an exponential series as

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} = F_0 + \sum_{\substack{n=-\infty \\ n \neq 1}}^{\infty} F_n e^{jn\omega_0 t} \quad --- (1.48)$$

where $$F_n = \frac{1}{T_0} \int_{t_1}^{t_1+T_0} f(t)\, e^{-jn\omega_0 t}\, dt \quad --- (1.49)$$

~~The exponential Fourier series is basically another~~ form of trigonometric Fourier series.
It is much more convenient to handle the exponential series than the trigonometric one. In system analysis also, the exponential form proves more convenient than the trigonometric form. For these reasons we shall use exponential (rather than trigonometric) representation of signals.

Ex. 1-24 Find the exponential Fourier series for the signal shown in Fig.(1.44) $f(t)\ e^{-t/2}$

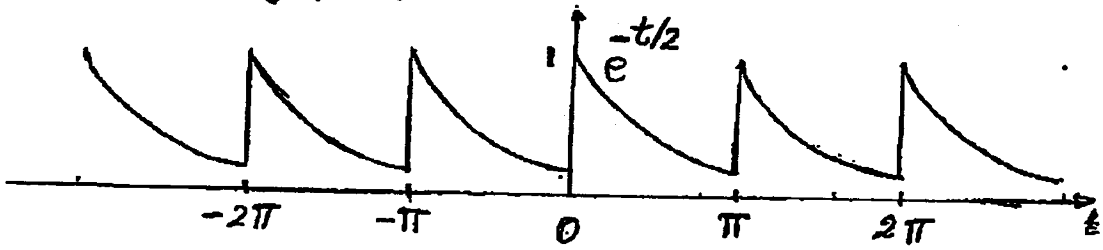


Fig.(1-44)

Solution:

in this case, $T_0 = \pi$, $\omega_0 = \frac{2\pi}{\pi} = 2$ and

$$f(t) = \sum_{n=-\infty}^{\infty} F_n\, e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} F_n\, e^{j2nt}$$

where

$$F_n = \frac{1}{T_0}\int_0^{T_0} f(t)\, e^{-jn\omega_0 t}\, dt = \frac{1}{\pi}\int_0^{\pi} e^{-t/2}\, e^{-j2nt}\, dt$$

$$= \frac{1}{\pi}\int_0^{\pi} e^{-(\frac{1}{2}+j2n)t}\, dt = \frac{1}{\pi}\left[\frac{-1}{\frac{1}{2}+j2n}\, e^{-(\frac{1}{2}+j2n)t}\right]_0^{\pi}$$

$$F_n = \frac{-2}{\pi(1+j4n)}\left(e^{-(\frac{1}{2}+j2n)\pi} - e^{0}\right)$$

$$= \frac{-2}{\pi(1+j4n)}\left(e^{-\pi/2}\, e^{-j2n\pi} - 1\right)$$

Note that $e^{-\pi/2} = 0.2078$

and $e^{-j2n\pi} = (\cos 2n\pi - j \sin 2n\pi)$

$\cos 2n\pi = 1$, $\sin 2n\pi = 0$ for any Value of n

$$\therefore F_n = \frac{-2}{\pi(1+j4n)}\left(0.2078-1\right) = \frac{0.504}{1+j4n} = 0.504\left(\frac{1-j4n}{1+16n^2}\right)$$

$F_n = |F_n|\, e^{j\theta_n}$ Where $|F_n| = \sqrt{Imag^2 + Real^2}$

$$|F_n| = 0.504\, \frac{1}{\sqrt{1+16n^2}}$$

and $\theta_n = \tan^{-1} \frac{Imag}{Real} = \tan^{-1} \frac{-4n}{1} = -\tan^{-1} 4n$

$$f(t) = \sum_{n=-\infty}^{\infty} F_n\, e^{j2nt} = 0.504 \sum_{n=-\infty}^{\infty} \frac{1-j4n}{1+16n^2}\, e^{j2nt}$$

$$f(t) = 0.504\Big(\underbrace{\frac{1+j4}{1+16}e^{-j2t}}_{\substack{F_{-1}\\ \text{for } n=-1}} + \underbrace{\frac{1+j8}{1+64}e^{-j4t}}_{\substack{F_{-2}\\ \text{for } n=-2}} + \underbrace{\frac{1+j12}{1+144}e^{-j6t}}_{\substack{F_{-3}\\ \text{for } n=-3}} + \cdots +$$

$$+ \underbrace{1}_{\substack{\text{for } n=0\\ F_0}} + \underbrace{\frac{1-j4}{1+16}e^{j2t}}_{\substack{\text{for } n=1\\ F_1}} + \underbrace{\frac{1-j8}{1+64}e^{j4t}}_{\substack{\text{for } n=2\\ F_2}} + \underbrace{\frac{1-j12}{1+144}e^{j6t}}_{\substack{\text{for } n=3\\ F_3}} \cdots\Big)$$

Note that the coefficients F_n are complex. Moreover, F_n and F_{-n} are conjugates.

Exponential Fourier spectra

In exponential spectra, the coefficients F_n are ploted as a function of $\omega = n\omega_0$. But since F_n is complex in general, we need two plots: the real and imaginary parts of F_n or we plot the amplitude (magnitude $|F_n|$) and the angle of F_n. We prefer the second one because of its close connection to the amplitude and phase of the trigonometric Fourier series. We therefore plot $|F_n|$ vs. ω and θ_n vs. ω. This requires that the coefficients F_n be expressed in polar form as

$$F_n = |F_n|\, e^{j\theta_n} \qquad (1.50)$$

~~It must be noted that~~

$$F_0 = a_0 = C_0 \qquad (1.51)$$

and for real periodic signal the coeffcients F_{+n} and F_{-n} are conjugates, that is

$$F_{+n} = F_{-n}^{*} \qquad (1.52a)$$

$$F_{+n}^{*} = F_{-n} \qquad (1.52b)$$

$$|F_{+n}| = |F_{-n}| = \frac{1}{2} C_n \qquad (1.52c)$$

If θ_{+n} is the phase of F_{+n}, then

θ_{-n} is the Phase of F_{-n}

Thus $F_{+n} = |F_{+n}| e^{j\theta_n}$ and $F_{-n} = |F_{-n}| e^{-j\theta_n} = |F_{+n}| e^{-j\theta_n}$ (1.53)

And when $f(t)$ is a real function, then the amplitud spectrum ($|F_n|$ vs. ω) is an <u>even function of ω</u>, and the angle spectrum (θ_n vs. ω) is an <u>odd funtion of ω</u>

Return to the series in Ex. 1.24

$$F_0 = 0.504$$

$$F_{-1} = \frac{0.504}{\sqrt{17}} e^{j75.96^\circ} = 0.122\, e^{j75.96^\circ}, \quad |F_{-1}| = 0.122$$

$$\cancel{F_1 = 0.504\,\frac{1-j4}{1+6} = \frac{0.504}{\sqrt{17}}\,e^{-j75.96^\circ} = 0.122\,e^{-j75.96^\circ}\;;\;|F_1| = 0.122}$$

$$F_{-2} = 0.504\,\frac{1+j8}{65} = \frac{0.504}{\sqrt{65}}\,e^{j82.87} = 0.0625\,e^{j82.87^\circ}\;,\;|F_{-2}| = 0.0625$$

$$F_2 = 0.504\,\frac{1-j8}{65} = \frac{0.504}{\sqrt{65}}\,e^{-j82.87^\circ} = 0.0625\,e^{-j82.87}\;;\;|F_2| = 0.0625$$

$$F_{-3} = 0.504\,\frac{1+j12}{145} = \frac{0.504}{\sqrt{145}}\,e^{j85.23} = 0.0418\,e^{j85.23}\;,\;|F_{-3}| = 0.0418$$

$$F_3 = 0.504\,\frac{1-j12}{145} = \frac{0.504}{\sqrt{145}}\,e^{-j85.23} = 0.0418\,e^{-j85.23}\;,\;|F_3| = 0.0418$$

and so on. Note that F_n and F_{-n} are conjugates as expected. Fig.(1-45) shows the frequency spectra (amplitude and phase) of the exponential Fourie series for the periodic signal $e^{-t/2}$ given in Fig. 1-44

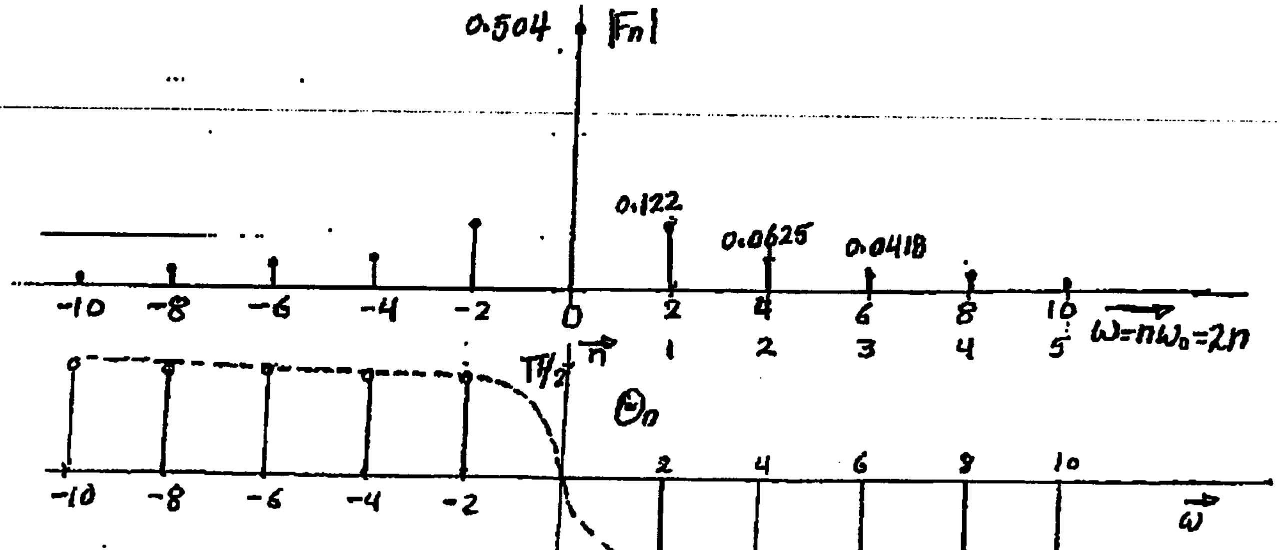
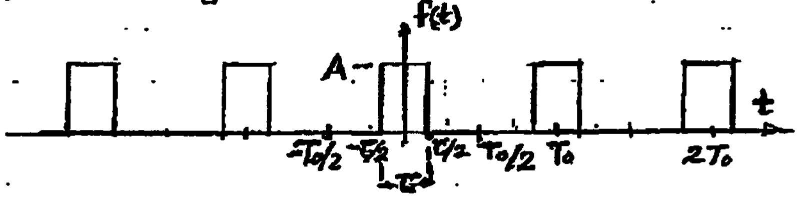


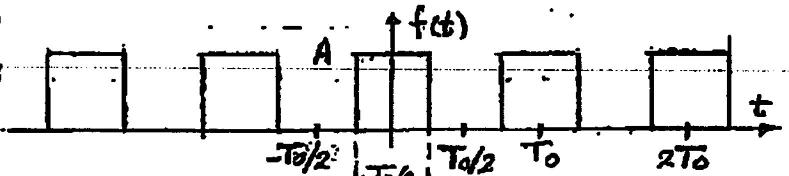
Fig.(1.44)

~~Let us consider a periodic gate function shown in~~

Fig. (1.45a). The Periodic gate function is a generlized Case of the Square wave signal shown in Fig.(1.45b).
A Periodic gate function ~~is a rectangular pulses~~ with hight (A), width (τ) and periode T_0.
Gate function and its Corresponding spectrum play a major role in the " of may systems.



(a) Periodic gate function



(b) Square-wave signal

Fig. (1.45)

<u>Ex. 1-23</u>

Find the Fourier Spectrum of the Periodic gate function shown in Fig.(1.46)

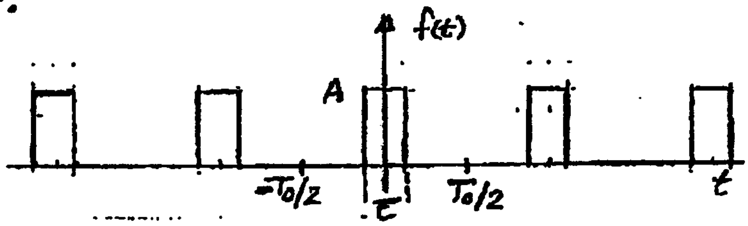


Fig. (1-46)

<u>Solution</u>:

$$F_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} f(t)\, e^{-jn\omega_0 t}\, dt = \frac{1}{T_0} \int_{-\tau/2}^{\tau/2} A\, e^{-jn\omega_0 t}\, dt$$

$$F_n = \frac{-A}{T_0\, n\omega_0'}\, e^{-jn\omega_0 t}\Big|_{-\tau/2}^{\tau/2} = \frac{-A}{jn\omega_0 T_0}\left(e^{-jn\omega_0\tau/2} - e^{+jn\omega_0\tau/2}\right), \quad n \neq 0$$

since $\sin\alpha = \frac{1}{2j}\left(e^{j\alpha} - e^{-j\alpha}\right)$

$$\therefore F_n = \frac{2A}{n\omega_0 T_0}\sin(n\omega_0\tau/2) \qquad n \neq 0$$

$$F_n = \frac{A\tau}{T_0}\,\frac{\sin n\omega_0\tau/2}{(n\omega_0\tau/2)} = \frac{A\tau}{T_0}\operatorname{Sinc}(n\omega_0\tau/2) \qquad n = 0$$

The function $\operatorname{Sinc}(x) = \frac{\sin(x)}{x}$ is show in Fig (1-47) for continous x. This function occurs so often in the comunication problems. From Fig. (1-47), the amplitude of the function $\operatorname{Sinc}(x)$ oscillates, decaying in either direction of x and approaching zero as $|x| \to \infty$. The maximum value of this function occurs as x approaches zero

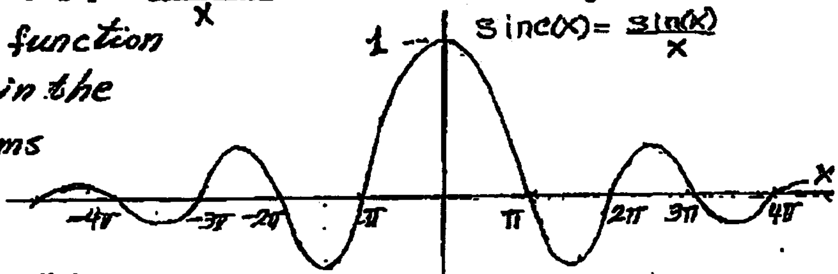


Fig.-(1- 47)

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

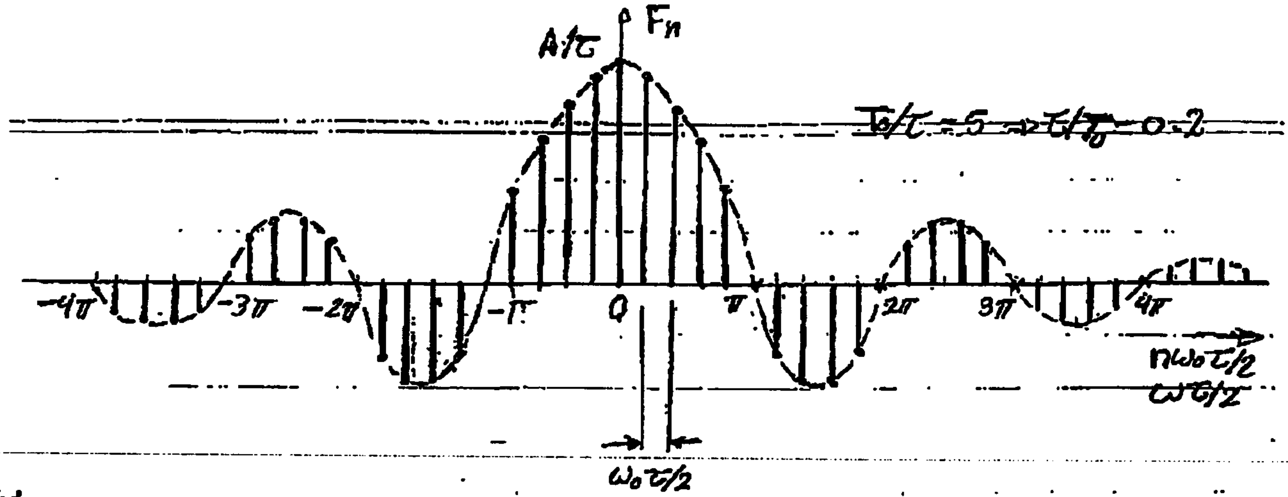
This canbe seen by using l' Hospital's rule

For $n = 0$

$$F_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} A\,dt = A\tau/T_0$$

$$\therefore f(t) = F_0 + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} F_n e^{jn\omega_0 t} = \frac{A\tau}{T_0} + \sum_{\substack{n=-\infty \\ n=0}}^{\infty} \frac{A\tau}{T_0}\operatorname{Sinc}(n\omega_0\tau/2)\,e^{jn\omega_0 t}$$

$$f(t) = \frac{A\tau}{T_0}\sum_{n=-\infty}^{\infty}\operatorname{Sinc}(n\omega_0\tau/2)\,e^{jn\omega_0 t} = \frac{A\tau}{T_0}\sum_{n=-\infty}^{\infty}\operatorname{Sinc}(n\pi\tau/T_0)\,e^{jn\omega_0 t}$$



The zero-crossing are the points where $F_n = 0$
where $\omega_0 = 2\pi/T_0$

Since F_n's are real, we need only display the amplitude spectrum. Aside from the scaling constant A, this spectrum is dependent only on the parametr T_0/τ or τ/T_0

For $\text{Sinc}(n\omega_0\tau/2)$ the variable $n\omega_0\tau/2$ is discrete. The spacing between two successive spectral lines is

$$\omega_0\tau/2 = \frac{\tau}{T_0}\pi .$$

The number of spectral lines between two successive zero-corssing is

$$K = \frac{\pi}{\omega_0\tau/2} - 1 = \frac{T_0}{\tau} - 1$$

The most important zero-corossing point is the first one at π which determines the bandwidth of the signal (will explained in later chapters).

Chapter Two

Fourier Transform and Applications

Chapter two

Fourier transform and its applications

2.1 Introduction

We have seen in Fourier series how to decompose a signal in the time domain into its harmonic components when the signal is a periodic function that has a well-defined fundamental period T (and frequency $\omega = 2\pi/T$). This gives a discrete spectrum (set of frequencies).
But many signals when represented in the frequency domain have a continuous spectrum. These represent signals in the time domain that do not have a single fundamental frequency. That is they are not periodic functions.
Of course, in engineering, most signals are not periodic; e.g. the accelerations in earthquake, random vibrations, chaotic outputs of simple nonlinear circuits, freak waves such as tsunamis the Severn bore, etc.. . .

Fourier series are applicable only to periodic functions but non-periodic functions can also be decomposed into Fourier components - this process is called a Fourier Transform.
So, we need a technique to find the frequency content of an arbitrary function f(t). The only stipulation is that the signal contain a “finite amount of energy”.

Such functions could have a finite duration, e.g.

$$f(t) = \begin{cases} 1 & \text{for } -1 < t < 1 \\ 0 & \text{otherwise} \end{cases}$$

or have decaying tails

$$f(t) = e^{-|t|} .$$

We will motivate the idea of the Fourier Transform as the analogue of the Fourier series but with continuous frequencies ω rather than discrete frequencies $\omega_n = 2\pi n/T$.

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

Definition: We say that F(ω) is the Fourier transform of f(t) and f(t) is the inverse Fourier transform of F(ω).
Sometimes f(t) and F(ω) are called a Fourier transform pair and written:

$$f(t) \longleftrightarrow F(\omega)$$

Example 1
Using the Fourier Transform integral equation, directly find the Fourier Transform of

$$x(t) = e^{-at}\ u(t),\ a > 0$$

solution

$$X(j\omega) = \int_0^{\infty} e^{-at}\, e^{-j\omega t}\,dt = \int_0^{\infty} e^{-(a+j\omega)t}\,dt$$

$$X(j\omega) = -\frac{e^{-at}e^{-j\omega t}}{a+j\omega}\Bigg|_0^{\infty} = \frac{1}{a+j\omega} \qquad \boxed{a>0}$$

$$\boxed{X(j\omega) = \frac{1}{a+j\omega}}$$

2.2 Properties of the Fourier transform ;

a. **Linearity**

If $f(t)$, $g(t)$ are functions with transforms $F(\omega)$, $G(\omega)$, respectively, then

- $F\{f(t) + g(t)\} = F(\omega) + G(\omega)$

Example;

1.
$$\begin{aligned}\mathcal{F}\{2e^{-t}u(t) + 3e^{-2t}u(t)\} &= \mathcal{F}\{2e^{-t}u(t)\} + \mathcal{F}\{3e^{-2t}u(t)\} \\ &= 2\mathcal{F}\{e^{-t}u(t)\} + 3\mathcal{F}\{e^{-2t}u(t)\} \\ &= \frac{2}{1+\mathrm{i}\omega} + \frac{3}{2+\mathrm{i}\omega}\end{aligned}$$

2. If
$$f(t) = \begin{cases} 4 & -3 \le t \le 3 \\ 0 & \text{otherwise} \end{cases}$$

b. Shift properties

There are two basic shift properties of the Fourier Transform:
(i) Time shift property:

$$\mathcal{F}\{f(t-t_0)\} = e^{-\mathrm{i}\omega t_0}F(\omega)$$

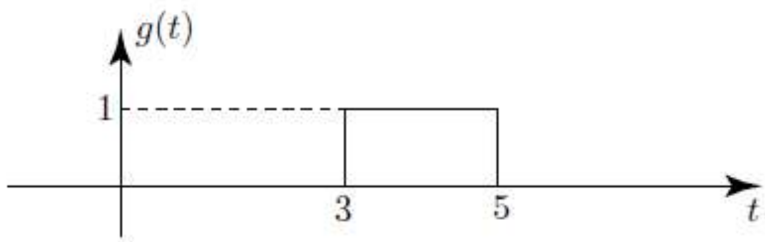
(ii) Frequency shift property

$$\mathcal{F}\{e^{\mathrm{i}\omega_0 t}f(t)\} = F(\omega - \omega_0).$$

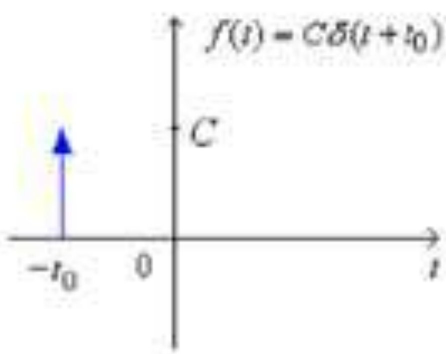
Here t_0, ω_0 are constants. In words, shifting (or translating) a function in one domain corresponds to a multiplication by a complex exponential function in the other domain.
We omit the proofs of these properties which follow from the definition of the Fourier Transform.

Example Use the time-shifting property to find the Fourier Transform of the function

$$g(t) = \begin{cases} 1 & 3 \le t \le 5 \\ 0 & \text{otherwise} \end{cases}$$



Example
Compute the Fourier transform of f(t)

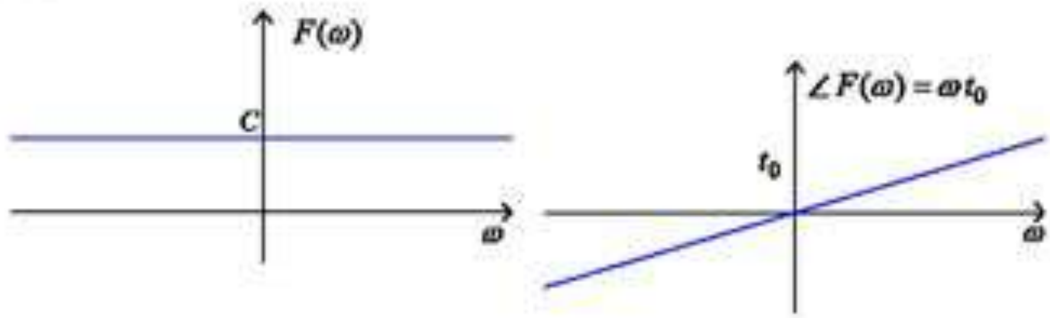


Solution

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} C\delta(t+t_o) e^{j\omega t} dt$$

Step 2. Evaluate the Integral

$$= Ce^{-j\omega t_o}$$



A more compact notation

In many applications you will find that a more compact notation is used for the Fourier series. Using the identity

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

We can write;

$$\cos(\theta) = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$

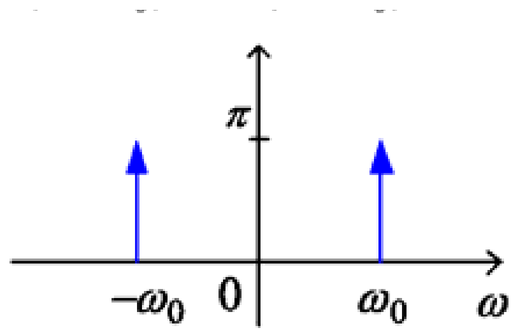
$$\sin(\theta) = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

Example
Find the Fourier Transform of cos(*ω*0*t*)
Solution ;

$$F[\cos(\omega_0 t)] = F[\frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})] \quad \text{where } C_1 = \frac{1}{2}, C_{-1} = \frac{1}{2}$$

$$F[\cos(\omega_0 t)] = \frac{1}{2}[2\pi\,\delta(\omega - \omega_0)] + \frac{1}{2}[2\pi\,\delta(\omega + \omega_0)]$$

$$= \pi\,\delta(\omega - \omega_0) + \pi\,\delta(\omega + \omega_0)$$



2.3 Inverse Fourier Transform:

It is always possible to move back from the frequency-domain to time-domain, by either summing the terms of the Fourier Series or by Inverse Fourier Transform

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$$

example
Find the Inverse Fourier Transform of the following function

$$\boxed{X(j\omega) = \begin{cases} 1 & |\omega| < \omega_b \\ 0 & |\omega| > \omega_b \end{cases}}$$

Solution

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\omega_b}^{\omega_b} 1\, e^{j\omega t}d\omega$$

$$x(t) = \frac{1}{2\pi}\frac{e^{j\omega t}}{jt}\Bigg|_{-\omega_b}^{\omega_b} = \frac{1}{2\pi}\frac{e^{j\omega_b t} - e^{-j\omega_b t}}{jt}$$

$$\boxed{x(t) = \frac{\sin(\omega_b t)}{\pi t}}$$

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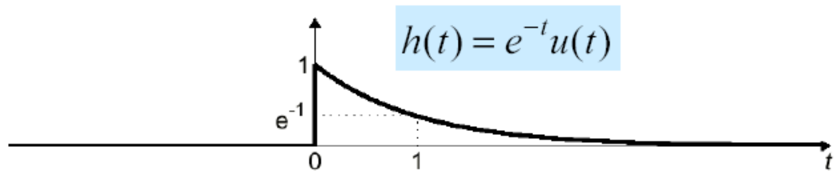
Frequency response

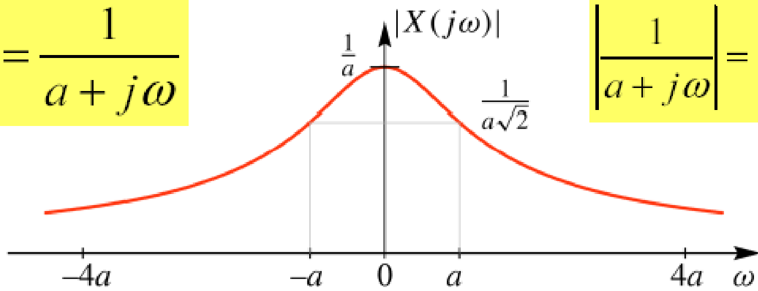
- Fourier Transform of h(t) **is** the Frequency Response



$$\boxed{h(t) = e^{-t}u(t) \Leftrightarrow H(j\omega) = \frac{1}{1 + j\omega}}$$

Magnitude and phase plots

$$H(j\omega) = \frac{1}{a + j\omega}$$

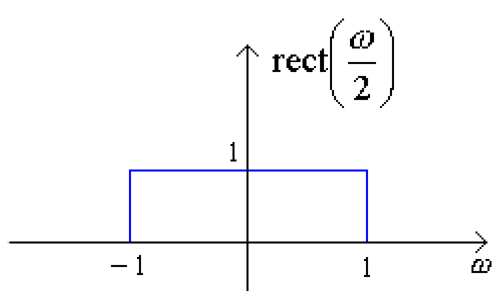


$$\left|\frac{1}{a + j\omega}\right| = \left|\frac{1}{\sqrt{a^2 + \omega^2}}\right|$$

Example

Find the inverse Fourier Transform of

$$X(w) = \begin{cases} 1 & -1 \leq w \leq 1 \\ 0 & \text{otherwise} \end{cases}$$



Solution

$$x(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi}\int_{-1}^{1} 1\, e^{j\omega t} d\omega = \frac{1}{2\pi}\frac{1}{jt} e^{\ j\omega t}\Big|_{-1}^{1}$$

$$= \frac{1}{2\pi jt}\left(e^{jt} \ - e^{-jt}\ \right) = \ \frac{\sin(\,t\,)}{\pi\, t}$$

Step 5. Simplify to the sinc function

$$= \frac{\omega_b}{\pi}\ \text{sinc}(\ \ t\ \ )$$

Solved Problem

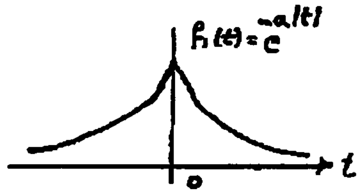
Ex.1. Find the Fourier transform of $f_1(t)$, shown in Fig. The function $f_1(t)$ is given by.

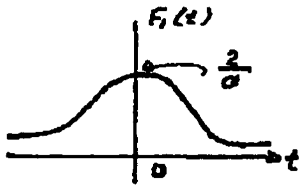
$$f_1(t) = \begin{cases} e^{at} & t<0 \\ e^{-at} & t>0 \end{cases}$$

solution

$$F_1(f) = \int_{-\infty}^{0} e^{at} e^{-j\omega t} + \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt$$

$$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{2a}{a^2+\omega^2}$$





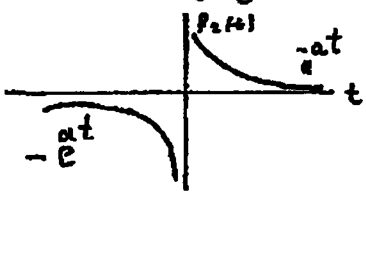
Ex2. Find the Fourier transform of $f_2(t)$ shown in fig. The function $f_2(t)$ is given by

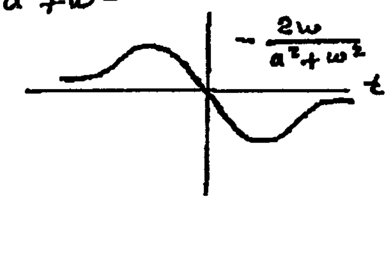
$$f_2(t) = \begin{cases} e^{-at} & t>0 \\ -e^{at} & t<0 \end{cases}$$

solution:

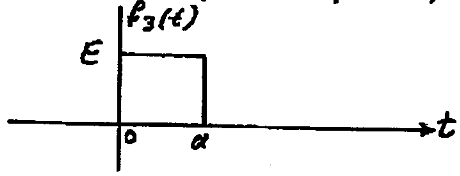
$$F_2(f) = \int_{-\infty}^{0} -e^{at} e^{-j\omega t} dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt$$

$$= \frac{-1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{-j2\omega}{a^2+\omega^2}$$





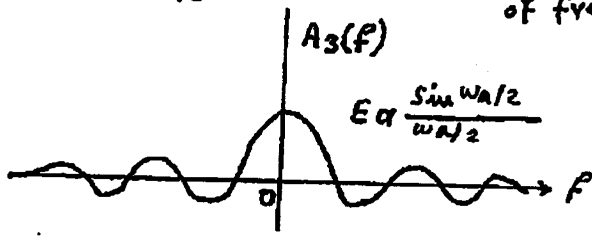
ex3. Find the Fourier transform of the square pulse shown in fig.

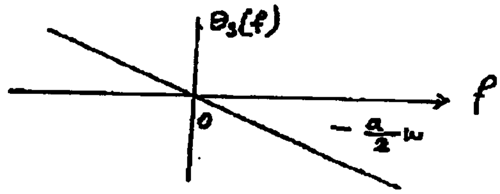


solution:

$$F_3(f) = \int_0^a E\, e^{-Jwt}\, dt = \frac{E}{Jw}\left(1 - e^{-Jwa}\right)$$

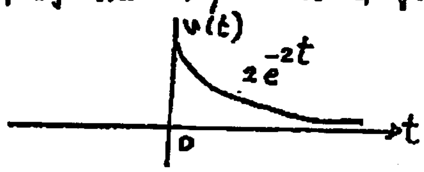
$$F_3(f) = E a \left(\frac{\sin wa/2}{wa/2}\right) e^{-Jwa/2}$$ which is a complex fun of frequency.





where $A_3(f)$ – the magnitude
$\theta_3(f)$ – the angle.

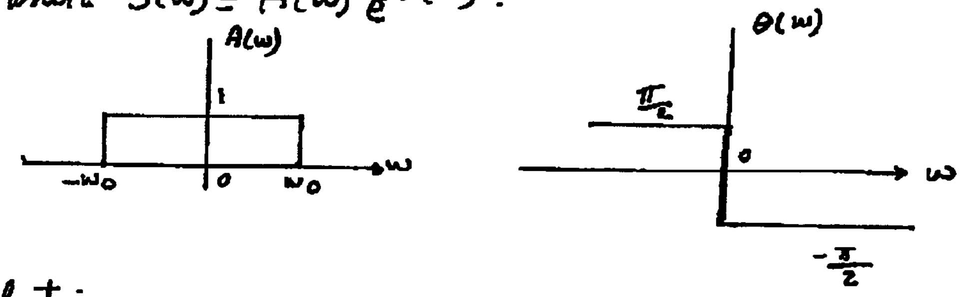
Ex4. Find the Fourier transform of the exponential function shown in fig.



solution:

$$V(f) = \frac{2}{2 + Jw}$$

EX.5 Find $s(t)$ corresponding to $S(w)$ shown in Fig.
where $S(w) = A(w)\, e^{j\theta(w)}$.



solution

$$s(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(w)\, e^{j\theta(w)}\, e^{jwt}\, dw$$

$$= \frac{1}{2\pi}\int_{-w_0}^{0} e^{j\frac{\pi}{2}}\, e^{jwt}\, dw + \frac{1}{2\pi}\int_{0}^{w_0} e^{-j\frac{\pi}{2}}\, e^{jwt}\, dw$$

$$= \frac{1}{\pi t}\left[\sin\frac{\pi}{2} + \sin\left(w_0 t - \frac{\pi}{2}\right)\right]$$

EX.6 Find the F.T of the unit gate function

$$f(t) = \mathrm{rect}(t) = \begin{cases} 1 & |t| < \frac{1}{2} \\ 0 & |t| > \frac{1}{2} \end{cases}$$

solution :

$$F(w) = \int_{-\infty}^{\infty} f(t)\, e^{-jwt}\, dt = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-jwt}\, dt$$

$$= \frac{-1}{jw}\, e^{-jwt}\Big]_{-1/2}^{1/2} = \frac{-1}{jw}\left(e^{-j\frac{w}{2}} - e^{j\frac{w}{2}}\right)$$

$$= \frac{\sin(w/2)}{(w/2)}$$

2.2 convolution theorem;

Commonly used in engineering, science, mathematically equal to

$$f_1(t)*f_2(t) \equiv \int_{-\infty}^{\infty} f_1(\tau)f_2(t-\tau)d\tau$$

To solve the equation above we do the following steps;

-Rotating one of the functions about the *y* axis

-Shifting it by *t*

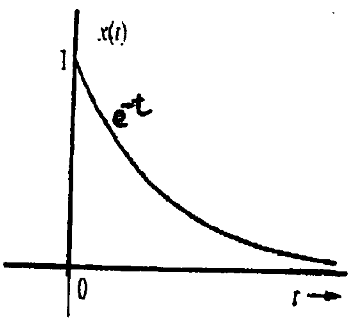
-Multiplying this flipped, shifted function with the other function

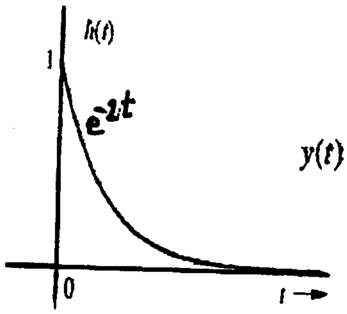
-Calculating the area under this product

-Assigning this value to $f_1(t) * f_2(t)$ at *t*

Example; 1

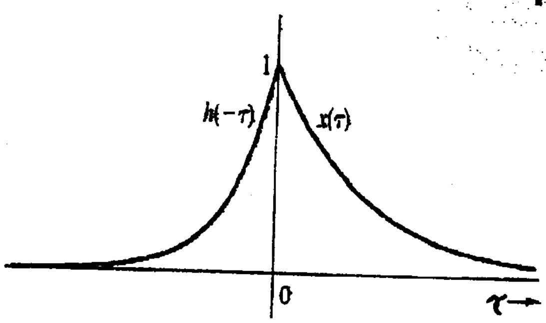
Determine graphically $y(t) = x(t)*h(t)$ **for** $x(t) = e^{-t}u(t)$ **and** $h(t) = e^{-2t}u(t)$.





$$y(t) = \int_0^t x(\tau)h(t-\tau)\,d\tau \qquad t \geq 0$$

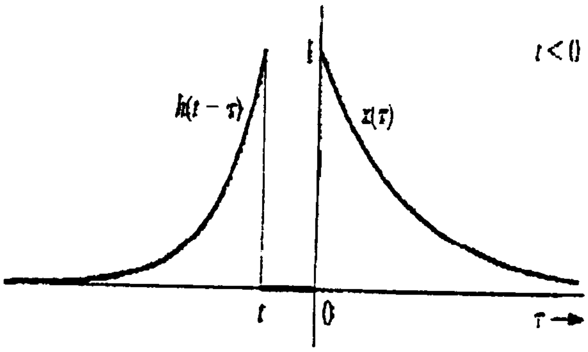
Remember: variable of integration is τ, not *t*

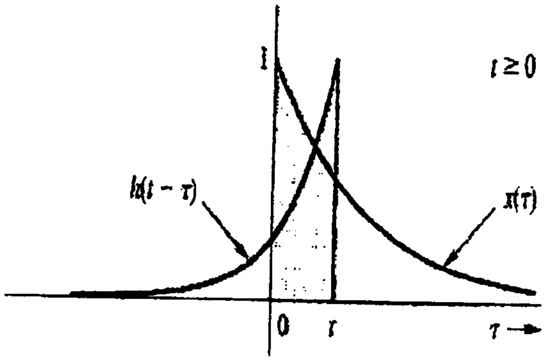


$$y(t) = \int_0^t e^{-\tau} e^{-2(t-\tau)}\, d\tau$$

$$= e^{-2t} \int_0^t e^{\tau}\, d\tau$$

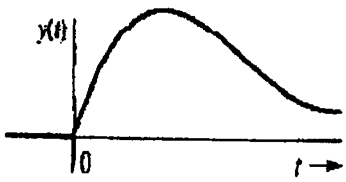
$$= e^{-t} - e^{-2t}$$





Moreover, $y(t) = 0$ for $t < 0$, so that

$$y(t) = (e^{-t} - e^{-2t})u(t)$$



Ex 2: Convolve the two step functions shown in Fig. 3

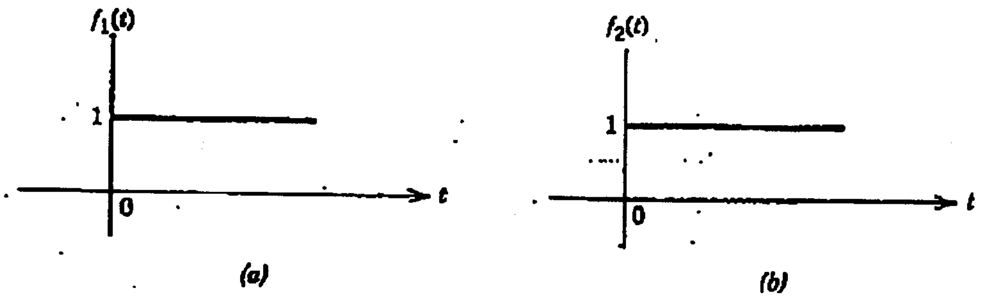


Figure 3 Two step functions to be convolved.

Solution

Since $f_1(t)$ and $f_2(t)$ are identical, we will arbitrarily choose to use Eq. 11b. Therefore $f_1(\lambda)$ and $f_2(t-\lambda)$ must be found. To find $f_1(\lambda)$ is easy; we simply replace t by λ in Fig. 3 a.

To find $f_2(t-\lambda)$, we can write

$$f_2(t) = \begin{cases} 1 & t>0 \\ 0 & t<0 \end{cases}$$

Then substitute $(t-\lambda)$ for t to get

$$f_2(t-\lambda) = \begin{cases} 1 & t-\lambda>0 \\ 0 & t-\lambda<0 \end{cases}$$

A preferred form is

$$f_2(t-\lambda) = \begin{cases} 1 & t>\lambda \\ 0 & t<\lambda \end{cases}$$

This same result can be obtained graphically. We begin in Fig. 4.a by plotting $f_2(\lambda)$ versus λ. This function is "flipped" in Fig. 4. b to obtain

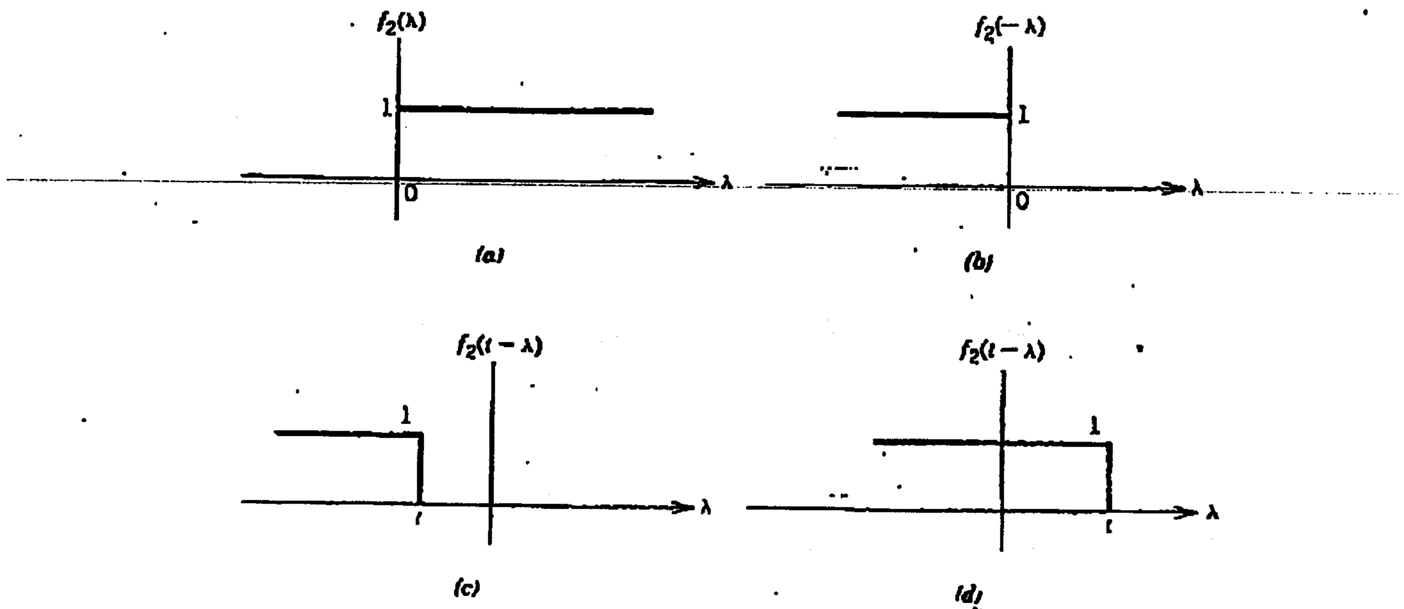


Figure. 4 Flipping and slipping f_2.

$f_2(-\lambda)$. Note that $f_2(-\lambda) = f_2(0-\lambda)$, or this is $f_2(t-\lambda)$ if $t=0$. This same function is shown for other values of t in Figs. 4 c and Fig. 4 d, first for a negative value of t (say, $t=-1$) and then for a positive value of t. This completes steps 1 and 2.

Now we must multiply $f_1(\lambda)$ by $f_2(t-\lambda)$, step 3, and integrate according to Eq. 11. This must be done for every value of t in the interval $-\infty < t < \infty$ (step 4). The solution is continued in Fig. 5, where in Fig. 5 a the value of t is less than zero and the product $f_1(\lambda)f_2(t-\lambda)$ is zero for every value of λ. In Fig. 5 b with $t>0$, the product $f_1(\lambda)f_2(t-\lambda)$ is equal to 1 for $0<\lambda<t$ and zero elsewhere. Thus $f_3(t)$ is given by

$$f_3(t) = \int_{-\infty}^{0} 0\,d\lambda + \int_0^t 1\,d\lambda + \int_t^{\infty} 0\,d\lambda = t, \qquad t>0$$

This is plotted in Fig. 5 c and we see that the convolution of two unit steps results in the unit ramp.

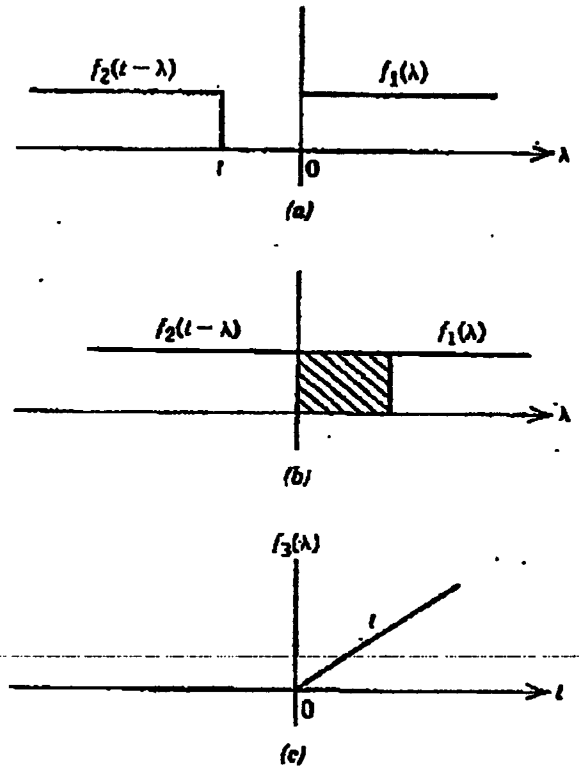


Figure 5. The process of convolving two step functions.

EX3.

Convolve the two functions shown in Fig. 6

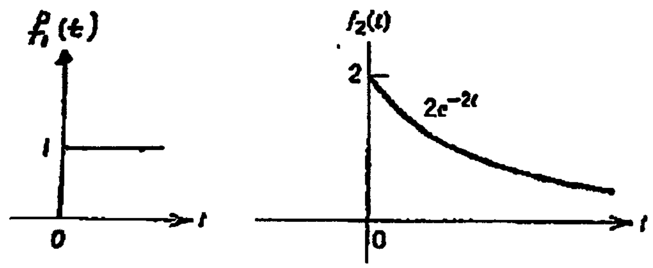


Figure 6. A step and an exponential function to be convolved.

Solution

We will graphically flip and slip $f_2(t)$ to illustrate how this is done. [It would be easier to flip and slip $f_1(t)$ since it is a simpler function than $f_2(t)$ and we would normally choose the easier approach.] Therefore, the following formula will be used.

$$f_3(t) = \int_{-\infty}^{\infty} f_1(\lambda) f_2(t-\lambda)\, d\lambda$$

Study Fig. 7. We proceed from $f_2(\lambda)$ in Fig. 7.*a* to $f_2(-\lambda)$ in Fig. 7.*b* to $f_2(t-\lambda)$ in Figs. 7.*c* and 7*d*. To evaluate $f_3(t)$ we note that $f_1(\lambda)f_2(t-\lambda)$ is zero for all $t<0$. Therefore, $f_3(t)$ is zero for all $t<0$. Now for $t>0$ we have (Fig. 7.*d*)

$$f_3(t) = \int_0^t 2e^{-2(t-\lambda)}\, d\lambda = 2e^{-2t}\int_0^t e^{-2\lambda}\, d\lambda = 1 - e^{-2t}, \qquad t>0$$

$f_3(t)$ is plotted in Fig. 7.*e*.

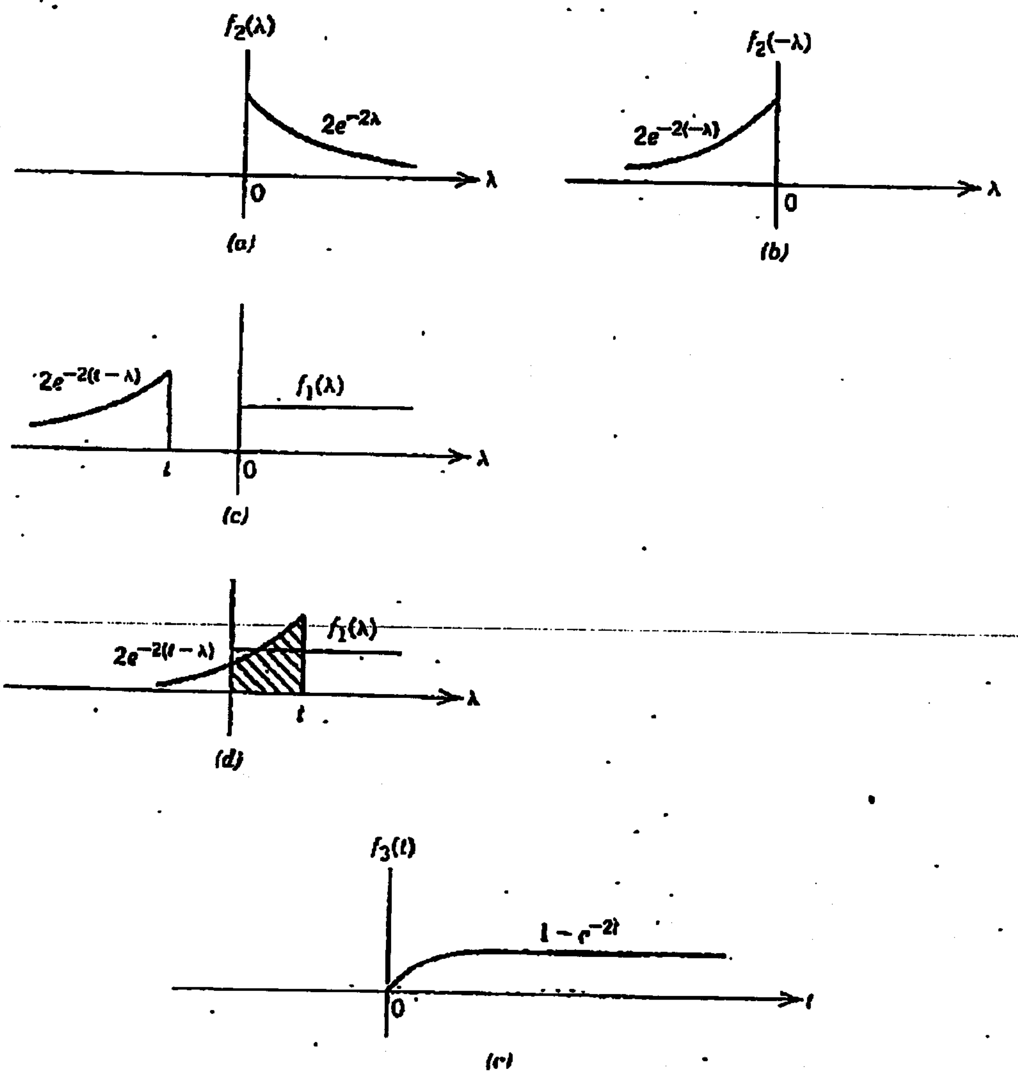


Figure 7. The convolution of a step and exponential function.

2.4 Power and Energy Spectral Density

- The power spectral density (PSD) $S_x(w)$ for a signal is a measure of its power distribution as a function of frequency
- It is a useful concept which allows us to determine the bandwidth required of a transmission system

- Consider a signal *x(t)* with Fourier Transform (FT) *X(w)*

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

- We wish to find the energy and power distribution of *x(t)* as a function of frequency

Consider an energy signal g(t), Parseval's Theorem states that

$$E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(\omega)|^2 d\omega$$

Proof:

$$\begin{aligned} E_g &= \int_{-\infty}^{\infty} g(t) g^*(t) dt = \int_{-\infty}^{\infty} g(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} G^*(\omega) e^{-j\omega t} d\omega \right] dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} G^*(\omega) \left[\int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt \right] d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) G^*(\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(\omega)|^2 d\omega \end{aligned}$$

2.4.1 Energy Spectral Density

• Parseval's theorem can be interpreted to mean that the energy of a signal g(t) is the result of energies contributed by all spectral components of a signal g(t).
• The contribution of a spectral component of frequency w is proportional to |G(w)|^2.
• Therefore, we can interpret |G(w)|^2 as the energy per unit bandwidth of the spectral components of g(t) centered at frequency w.
• In other words, |G(w)|^2 is the energy spectral density of g(t)

Example

Consider the signal $g(t) = e^{-at}u(t)$ $(a > 0)$.
Its energy is

$$E_g = \int_{-\infty}^{\infty} g^2(t)dt = \int_{0}^{\infty} e^{-2at}dt = \frac{1}{2a}$$

We now determine E_g using the signal spectrum $G(\omega)$ given by

$$G(\omega) = \frac{1}{j\omega + a}$$

It follows

$$E_g = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\omega^2 + a^2} d\omega = \frac{1}{2\pi a}[tan^{-1}\frac{\omega}{a}]_{-\infty}^{\infty} = \frac{1}{2a}$$

which verifies Parseval's theorem.

If *x(t)* is the voltage across a *R*=1 resistor, the instantaneous power is,

$$\frac{(x(t))^2}{R} = (x(t))^2$$

Thus the total energy in *x(t)* is,

$$\text{Energy} = \int_{-\infty}^{\infty} x(t)^2\, dt$$

From Parseval's Theorem,

$$\text{Energy} = \int_{-\infty}^{\infty} |X(\omega)|^2 \, df$$

$$= \int_{-\infty}^{\infty} |X(2\pi f)|^2 \, df$$

$$= \int_{-\infty}^{\infty} E(2\pi f) \, df$$

2.4.2 Power Spectral Density

The power P_g of a real signal $g(t)$ is given by

$$P_g = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} g^2(t) dt.$$

All the results for energy signals can be extended to power signals. Call $S_g(\omega)$ the Power Spectral Density (PSD) of $g(t)$. Thus,

$$P_g = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_g(\omega) d\omega.$$

We find the average power by averaging over time

$$\text{Average power} = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} (x_T(t))^2 \, dt$$

Where $x_T(t)$ is the same as $x(t)$, but truncated to zero outside the time window $-T/2$ to $T/2$

- Using Parseval as before we obtain,

$$\text{Average power} = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} (x_T(t))^2 \, dt$$

$$= \lim_{T \to \infty} \frac{1}{T} \int_{-\infty}^{\infty} |X_T(2\pi f)|^2 \, df$$

$$= \int_{-\infty}^{\infty} \lim_{T \to \infty} \frac{|X_T(2\pi f)|^2}{T} \, df$$

$$= \int_{-\infty}^{\infty} S_x(2\pi f) \, df$$

Where $S_x(w)$ is the Power Spectral Density (PSD)

$$S_x(\omega) = \lim_{T \to \infty} \frac{|X_T(\omega)|^2}{T}$$

Example

Evaluate the power, the spectral density and autocorrelation function of the signal

f (t) = Acos w0t where w0 = 2π/T. We have

Solution

$$P = \frac{1}{T} \int_0^T A^2 \cos^2 \omega_0 t \; dt = \frac{A^2}{T} \times \frac{1}{2} \int_0^T (\cos 2\omega_0 t + 1) dt = A^2/2.$$

The evaluation of the average power of a sinusoid is often needed. It is worth while remember in g that the average power of a sinusoid of amplitude A is simply A^2/2.

We also note that the Fourier series coefficients of the expansion

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

$$F_n = \begin{cases} A/2, & n = \pm 1 \\ 0, & otherwise \end{cases}$$

$$P = \overline{f^2(t)} = \sum |F_n|^2 = 2 \times A^2/4 = A^2/2$$

$$S_{ff}(\omega) = 2\pi \sum |F_n|^2 \delta(\omega - n\omega_0) = \pi \frac{A^2}{2} \{\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\}$$

$$P = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\pi A^2}{2} \{\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\} d\omega = A^2/2$$

$$r_{ff}(t) = |F_0|^2 + 2\sum_{1}^{\infty} |F_n|^2 \cos n\ \omega_0 t = (A^2/2)\cos\omega_0 t.$$

$$R_{ff}(j\omega) = \frac{\pi A^2}{2} \{\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\} = S_{ff}(\omega).$$

2.5 time –averaged noise representations;

In general, noise is random. Hence we cannot deterministically evaluate it. However, using its statistical properties, we can quantitatively analyze and hence design procedures to remove random noise

To do this, we need to know the distribution of the process that generates the noiseTypically, the noise is characterized in one of two ways, according to its distribution

.Uniform random noise -------- Noise distribution is uniform

. Gaussian random noise --------- Noise distribution is Gaussian

Noise is also characterized, according to how it is integrated into the signal

Additive noise – most noise is assumed to be additive, because it is easier to work with

$x(t) = s(t) + n(t)$ **Measured signal = true signal + noise**

☐ Multiplicative noise

$x(t) = s(t) \times n(t)$ **Measured signal = true signal * noise**

For the uniform random noise, the mean and the average power, calculated theoretically. Noise is only statistically characterized. We cannot discuss a single event at a certain time.

time average: Averaged quantity of a single system over a time interval.

(directly related to real experiment)

mean: $\overline{x(t)} = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)dt$

mean-square: $\overline{x(t)^2} = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)^2 dt$

variance: $\sigma_x^2 = \overline{x(t)^2} - \overline{x(t)}^2$

rms:

$$x_{\text{rms}} = \sqrt{\overline{x^2}} = \sqrt{\lim_{T\to\infty} \frac{1}{T}\int_0^T x^2(t)dt}$$

2.6 correlation function;

When the autocorrelation function is normalized by mean and variance, it is sometimes referred to as the **autocorrelation coefficient**[1]

Given a signal $f(t)$, the continuous autocorrelation $R_{ff}(\tau)$ is most often defined as the continuous cross-correlation integral of $f(t)$ with itself, at lag τ.

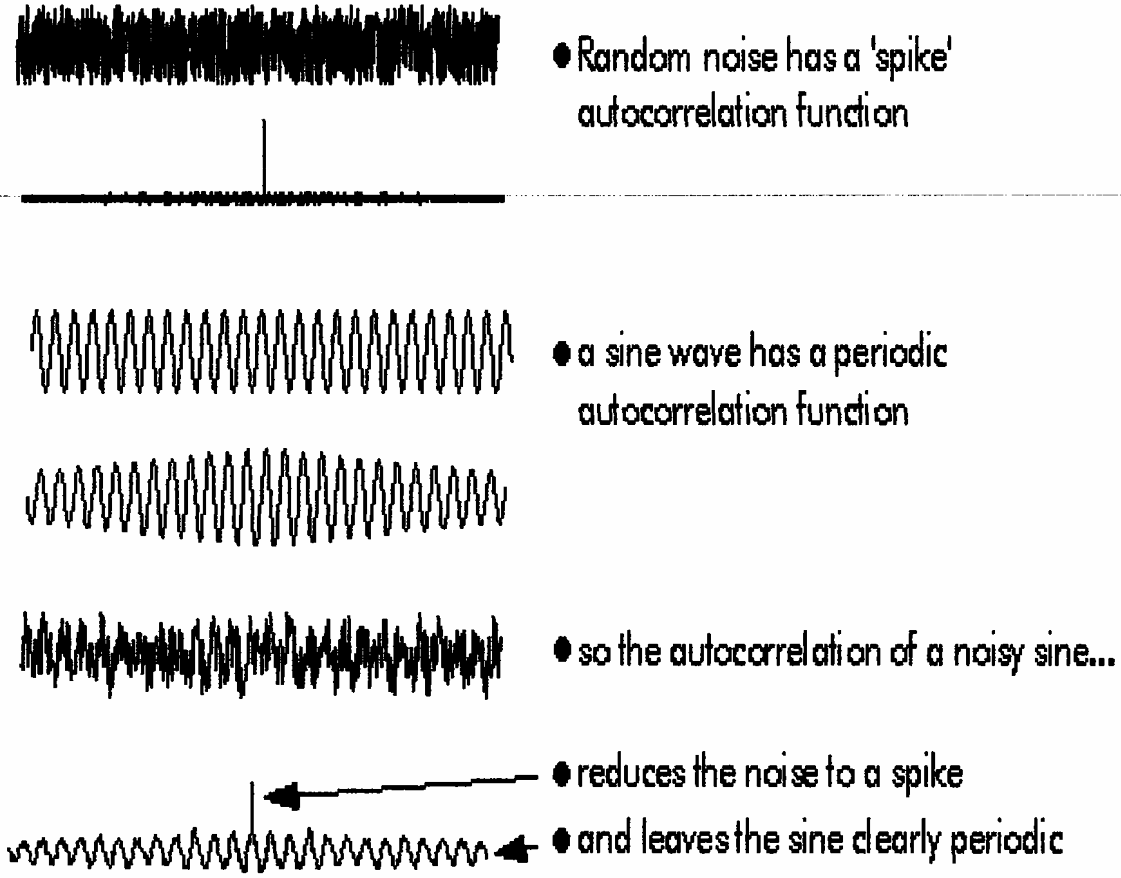
$$R_{ff}(\tau) = \overline{f}(-\tau) * f(\tau) = \int_{-\infty}^{\infty} f(t+\tau)\overline{f}(t)\,dt = \int_{-\infty}^{\infty} f(t)\overline{f}(t-\tau)\,dt$$

For processes that are also ergodic, the expectation can be replaced by the limit of a time average. The autocorrelation of an ergodic process is sometimes defined as or equated to

$$R_{ff}(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(t+\tau)\overline{f}(t)\,dt$$

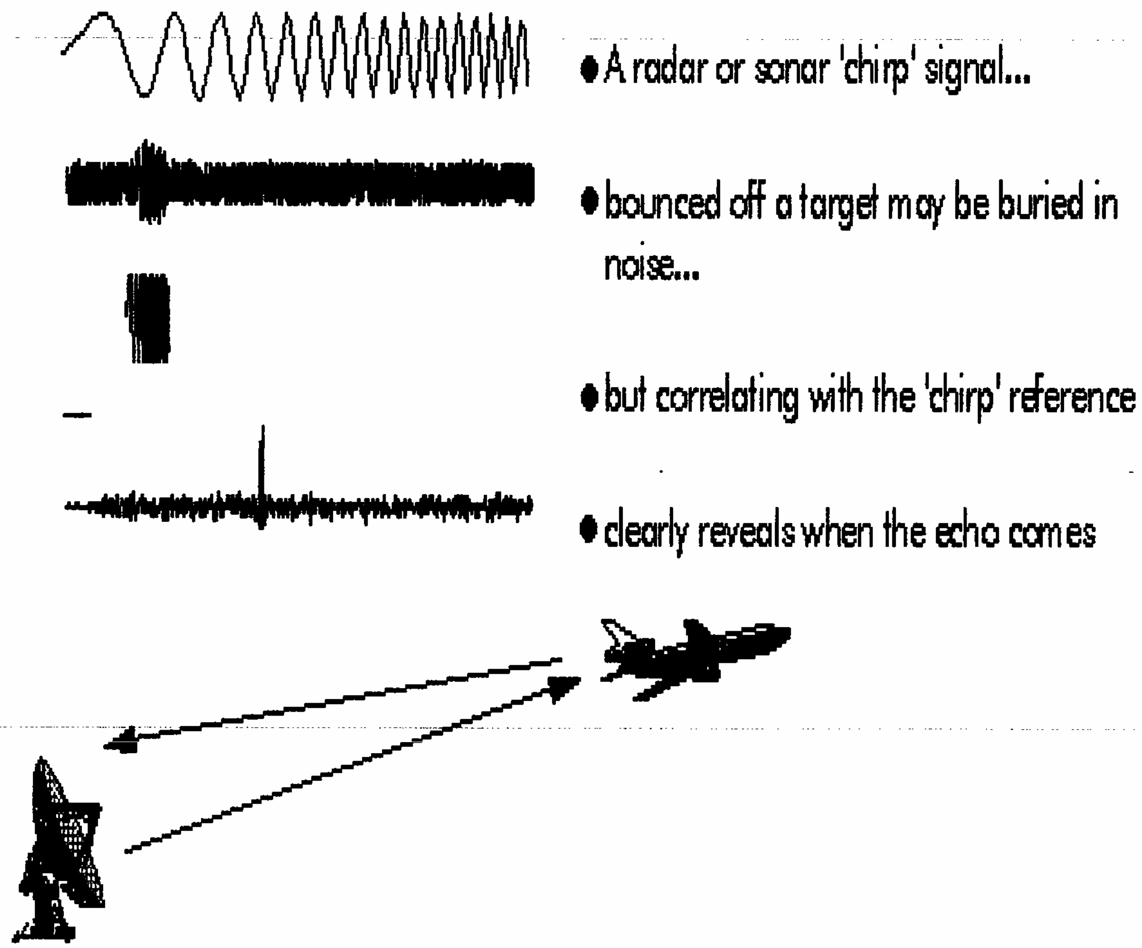
One way to characterize noise (or any signal) is to use the auto-correlation of the noise: The auto-correlation gives a measure of the "memory" of the system. A very important special case of correlation is *autocorrelation*. Autocorrelation is the correlation of a function with a shifted version of *itself*.

2.6.1 Autocorrelation can be used to extract a signal from noise

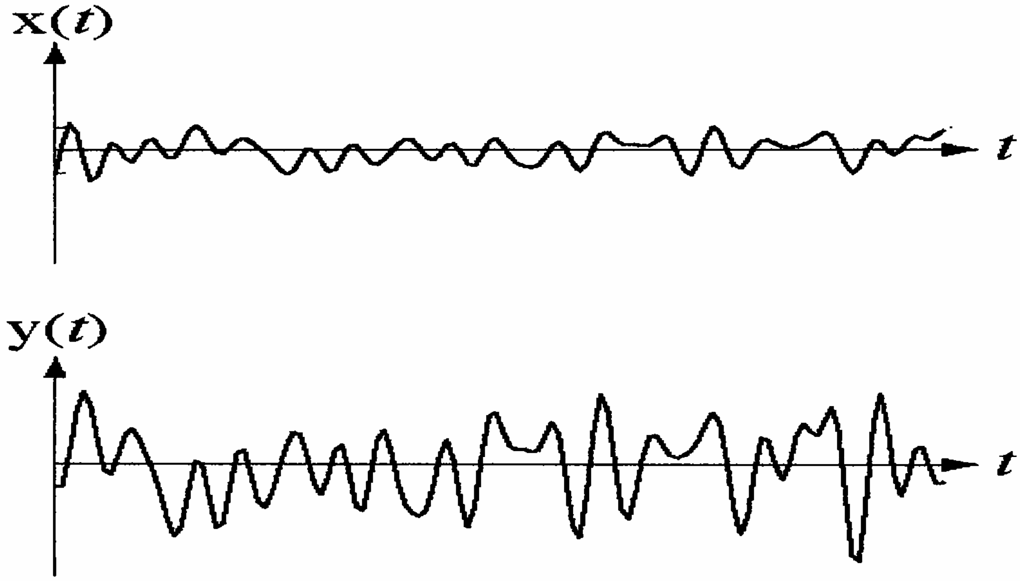


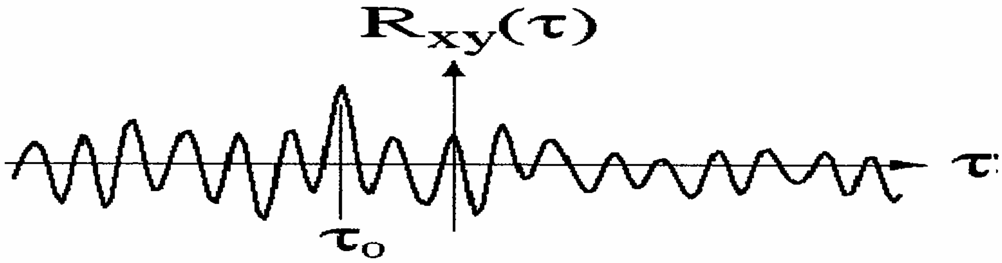
2.6.2 Cross correlation;

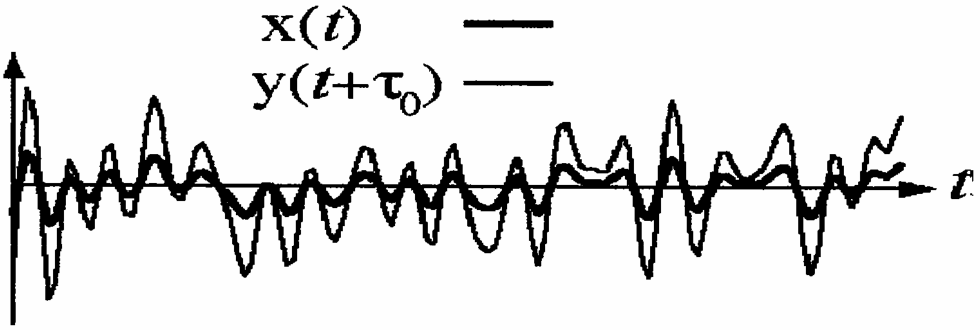
Cross correlation can be used to detect and locate known reference signal in noise. Cross correlation is really just "correlation" in the cases in which the two signals being compared are different. The name is commonly used to distinguish it from autocorrelation.



A comparison of x and y with y shifted for maximum correlation.







Example : 1

Calculate the (a) average value, (b) ac power and (c) r.m.s value of the periodic waveform

$$v(t) = 1 + \cos w_0 t .$$

Solution:

Becouse $v(t)$ is periodic, we can integrate over one period rather than taking the limit.

a) $$\overline{v(t)} = \frac{1}{T} \int_{-T/2}^{T/2} (1 + \cos w_0 t)\, dt = 1$$

b) $$\overline{\sigma^2(t)} = \frac{1}{T} \int_{-T/2}^{T/2} (\cos w_0 t)^2\, dt = \frac{1}{2}$$

c) $$\overline{v^2(t)} = \frac{1}{T} \int_{-\frac{T}{2}}^{T/2} (1 + \cos w_0 t)^2\, dt$$

$$= \frac{1}{T} \int_{-\frac{T}{2}}^{T/2} (1 + 2\cos w_0 t + \cos^2 w_0 t)\, dt = \frac{3}{2}$$

$$V_{r.m.s} = \sqrt{\overline{v^2(t)}} = \sqrt{\frac{3}{2}}$$

Example 2

Determine and sketch the autocorrelation function of a periodic square wave with peak-to-peak amplitude A, period T, and mean value $A/2$.

Solution Because $f(t)$ is periodic, the limiting operation in the determination of $R_f(\tau)$ can be replaced by a computation over one period. Using an alternate form of Eq. (3.33) [see footnote to that equation], we have

For $-T/2 < \tau < 0$:

$$R_f(\tau) = \frac{1}{T}\int_{-T/4}^{(T/4)+\tau} A^2\,dt = A^2\left(\frac{1}{2} + \frac{\tau}{T}\right)$$

† With a change of variable, this can also be written as

$$R_f(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} f^*(t-\tau)f(t)\,dt.$$

For $0 < \tau < T/2$:

$$R_f(\tau) = \frac{1}{T}\int_{\tau-T/4}^{T/4} A^2\,dt = A^2\left(\frac{1}{2} - \frac{\tau}{T}\right).$$

A graph of $f(t)$ and $R_f(\tau)$ is shown in Fig. 3.2. Since $f(t+T) = f(t)$, all calculations repeat over every period. It follows that the autocorrelation function of a periodic waveform is periodic. Similarly, the autocorrelation function of an aperiodic waveform is aperiodic.

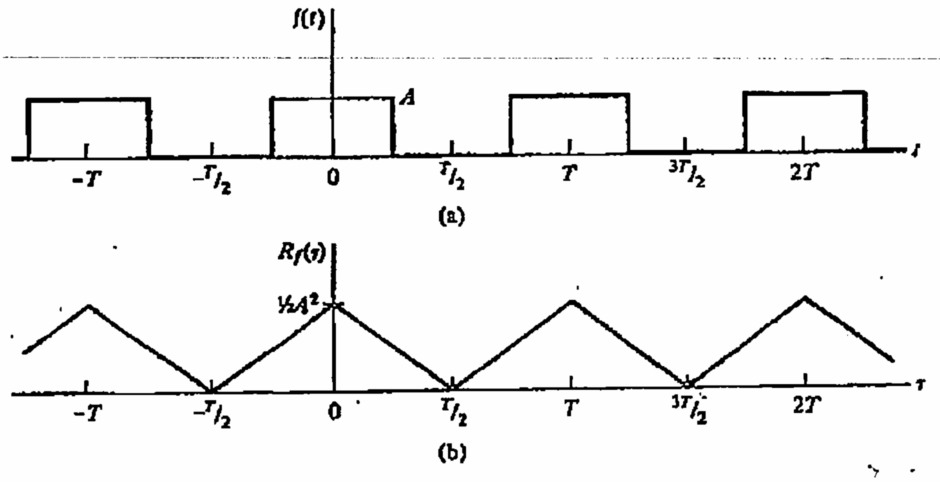


Figure 3.2 (a) A periodic waveform and (b) its autocorrelation function.

Example 3

Find the autocorrelation function of $\sqrt{2}\cos(\omega_0 t + \theta)$.

Solution

$$R_f(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} 2\cos(\omega_0 t + \theta)\cos(\omega_0 t + \omega_0\tau + \theta)\,dt,$$

$$R_f(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} \cos\omega_0\tau\,dt + \frac{1}{T}\int_{-T/2}^{T/2} \cos(2\omega_0 t + \omega_0\tau + 2\theta)\,dt,$$

$$R_f(\tau) = \cos\omega_0\tau.$$

Note that the autocorrelation function is independent of the phase θ.

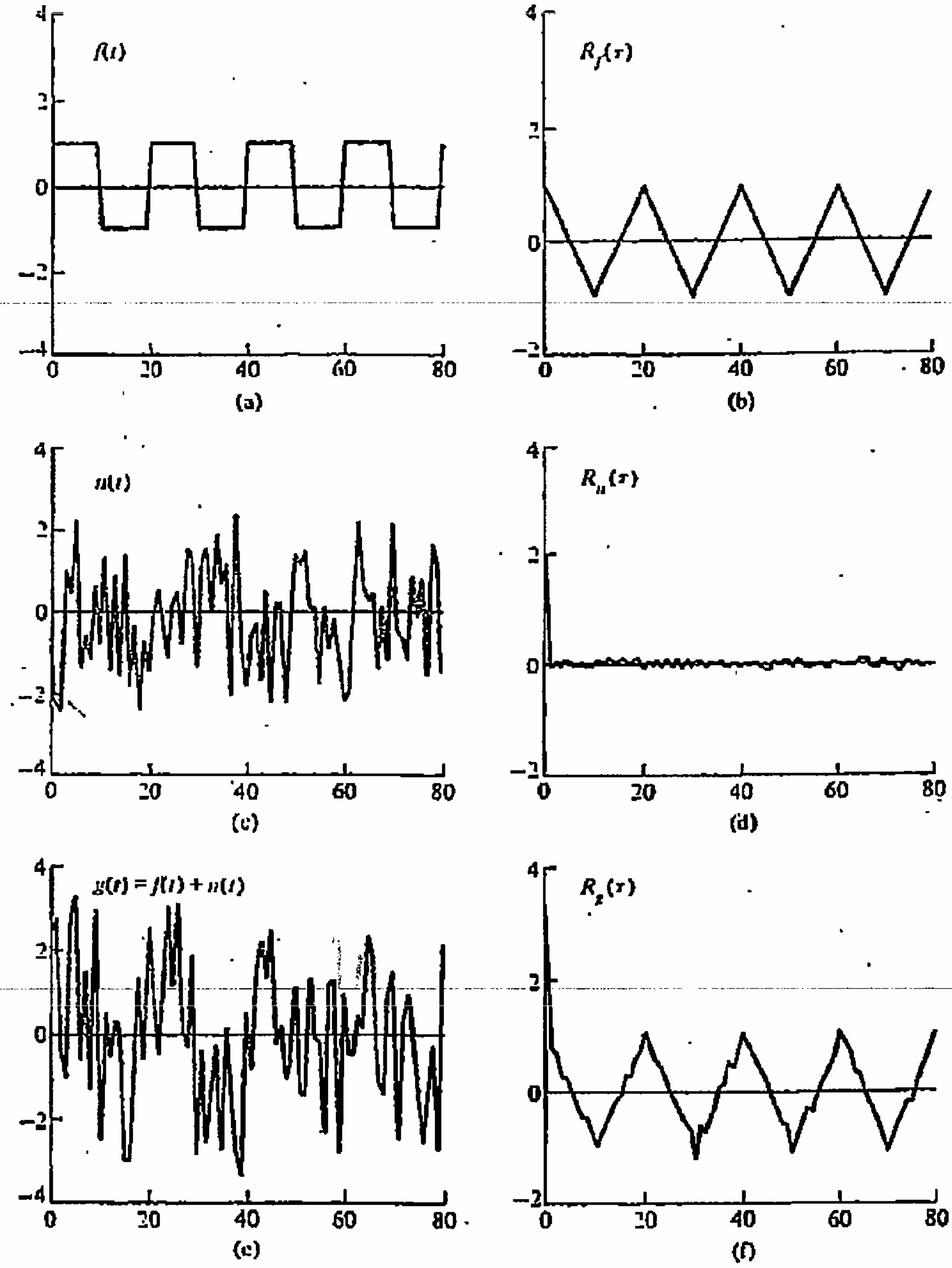


Figure 3.3 Autocorrelation of a periodic signal plus noise.